Three dimensional real Lie bialgebras

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Abstract

We classify all real three dimensional Lie bialgebras. In each case, their automorphism group as Lie bialgebras is also given.

Introduction

Our goal is to classify the *real* three dimensional Lie bialgebras. Recall that a Lie bialgebra over a field $\mathbb{K}$ is a triple $(\mathfrak{g}, [-,-], \delta)$ where $(\mathfrak{g}, [-,-])$ is a Lie algebra over $\mathbb{K}$ and $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is such that

- the induced map $\delta^* : \Lambda^2\mathfrak{g}^* \subseteq (\Lambda^2\mathfrak{g})^* \to \mathfrak{g}^*$ is a Lie algebra structure on $\mathfrak{g}^*$,
- $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle in the Chevalley-Eilenberg complex of the Lie algebra $(\mathfrak{g}, [-,-])$ with coefficients in $\Lambda^2\mathfrak{g}$.

The Jacobi condition for δ^* is called co-Jacobi condition. A Lie bialgebra is said factorizable, if there exist $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta(x) = \mathrm{ad}_x(r)$ $\forall x \in \mathfrak{g}$, r satisfies the classical Yang-Baxter equation and the symmetric component of r induces a nondegenerate inner product on $\mathfrak{g}^*$. According to [A-J], a real Lie bialgebra is said almost factorizable if the complexification is factorizable.

In [G], the author gives a classification of three dimensional Lie bialgebras, but for example, in the $\mathfrak{sl}(2,\mathbb{R})$ case, we find differences between his result and our. Namely, we find for $\mathfrak{sl}(2,\mathbb{R})$, three families of isomorphism classes of Lie bialgebras, apart from the co-abelian (see section 10 for the details). Our first case is a 1-parameter family: the cobracket is a positive multiple of $\mathrm{ad}(r)$ with $r = y \wedge x$ in the standard basis $\{h, x, y\}$. The multiple can always be chosen positive because we prove that $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, -\delta)$ give isomorphic Lie bialgebras in this case, while in [G] they appear as non isomorphic. Our second 1-parameter family coincides with the one in [G]. On the other hand, for the third type, $r_\pm = \pm\frac{1}{2}h \wedge x$ give two non-isomorphic (triangular) Lie bialgebras on $\mathfrak{sl}(2,\mathbb{R})$, unlike the situation in [G] where only one possibility of sign is found.

In [RA-H-R] all the r-matrices for real 3-dimensional Lie algebras are computed; but for 3-dimensional solvable Lie algebras $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ is not trivial. Besides, in our work, we distinguish the isomorphism classes of Lie bialgebras. Although we do not find explicitly the r-matrices in the cases of coboundary Lie bialgebras, it is not hard to compute them. We compute $(\Lambda^2\mathfrak{g})^\mathfrak{g}$ and all the 1-cocycles, which imply the computation of $\dim H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ (see table below), since the space of coboundaries is isomorphic to $\Lambda^2\mathfrak{g}/(\Lambda^2\mathfrak{g})^\mathfrak{g}$.

Dimension of $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ for real 3-dimensional Lie algebras

$\mathfrak{g}$	$\mathfrak{h}_3$	$\mathfrak{r}_3$	$\mathfrak{r}_{3,\lambda}$ $\lambda\neq\pm1$	$\mathfrak{r}_{3,\lambda}$ $\lambda=-1$	$\mathfrak{r}_{3,\lambda}$ $\lambda=1$	$\mathfrak{r}'_{3,\lambda}$ $\lambda\neq0$	$\mathfrak{r}'_{3,\lambda}$ $\lambda=0$	$\mathfrak{su}(2)$	$\mathfrak{sl}(2,\mathbb{R})$
$\dim(\Lambda^2\mathfrak{g})^\mathfrak{g}$	2	0	0	1	0	0	1	0	0
dim(1-coboundaries)	1	3	3	2	3	3	2	3	3
dim(1-cocycles)	6	4	4	4	6	4	4	3	3
$\dim(H^1(\mathfrak{g}, \Lambda^2\mathfrak{g}))$	5	1	1	2	3	1	2	0	0

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Our method is direct: we fix a Lie algebra structure, find all possible 1-cocycles, solve the co-Jacobi condition and let the Lie algebra automorphisms group act on the set of solutions; in this way we find simultaneously the isoclasses of Lie bialgebras and its automorphism group as Lie bialgebras. Our main result is the complete classification of the real 3-dimensional Lie bialgebras, which is given case by case in each section; the Lie bialgebras automorphisms groups are given as well.

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1 General results

The center. Given a Lie bialgebra $(\mathfrak{g},[-,-],\delta)$, if one fixes the Lie algebra structure and varies δ, the 1-cocycle condition can be viewed as a set of linear equations in the matrix coefficients of the cobracket. Anyway, in some cases, the following property simplifies computations:

Proposition 1.1. *Let $\mathfrak{g}$ be a Lie algebra and $\delta:\mathfrak{g}\to\Lambda^2\mathfrak{g}$ a 1-cocycle, then $\delta(\mathcal{Z}\mathfrak{g})\subseteq(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$.*

Proof. Let $z\in\mathcal{Z}\mathfrak{g}$ and $x\in\mathfrak{g}$ arbitrary, the 1-cocycle condition reads

$$\delta[x,z]=[\delta x,z]+[x,\delta z]$$

But $z\in\mathcal{Z}\mathfrak{g}$ implies $[x,z]=0$ for any x, and also $[z,\Lambda^2\mathfrak{g}]=0$, so we conclude $[x,\delta z]=0$, $\forall x\in\mathfrak{g}$, namely, $\delta z\in(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. $\square$

The above proposition will be useful when $\mathcal{Z}\mathfrak{g}$ is "big" and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ "small". So, it will be useful to start computing the center and the invariant part or $\Lambda^2\mathfrak{g}$.

The derived ideal $[\mathfrak{g},\mathfrak{g}]$. Recall that a coideal in a Lie bialgebra $\mathfrak{g}$ is a subspace $V\subseteq\mathfrak{g}$ such that $\delta V\subseteq V\wedge\mathfrak{g}$. Such a subspace occurs as kernel of a Lie coalgebra map. The 1-cocycle condition for δ implies the following:

Proposition 1.2. *Let $(\mathfrak{g},\delta)$ be a Lie bialgebra, then $[\mathfrak{g},\mathfrak{g}]$ is a coideal. In particular, the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ admits a unique Lie bialgebra structure such that the canonical projection is a Lie bialgebra map. Moreover, if $(\mathfrak{g},\delta_1)\cong(\mathfrak{g},\delta_2)$ as Lie bialgebras, then $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],\overline{\delta}_1)\cong(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],\overline{\delta}_2)$.*

Notice that the Lie algebra structure on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is trivial, so a Lie bialgebra structure on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is equivalent to an usual Lie algebra structure on $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$.

Lemma 1.3. *Let $\mathfrak{g}$ be a Lie algebra and $\psi:\mathfrak{g}\to\mathfrak{g}$ a Lie algebra automorphism, then ψ induces Lie algebra morphisms $\psi|_{[\mathfrak{g},\mathfrak{g}]}:[\mathfrak{g},\mathfrak{g}]\to[\mathfrak{g},\mathfrak{g}]$ and $\overline{\psi}:\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\to\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. The applications* $\mathrm{Aut}(\mathfrak{g})\to\mathrm{Aut}([\mathfrak{g},\mathfrak{g}])$ *and* $\mathrm{Aut}(\mathfrak{g})\to\mathrm{Aut}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ *defined by $\psi\mapsto\psi|_{[\mathfrak{g},\mathfrak{g}]}$ and $\psi\mapsto\overline{\psi}$ are group homomorphisms.*

Remark 1.4. *Proposition 1.2 says that by trivializing the bracket one gets a quotient Lie bialgebra. The dual statement of Proposition 1.2 is about a subobject of $\mathfrak{g}$ instead of a quotient:* Ker δ *is a Lie subalgebra (due to the 1-cocycle condition) and it is obviously maximal with respect to the property of having trivial cobracket. If $\mathfrak{g}_1\cong\mathfrak{g}_2$ are two isomorphic Lie bialgebras, then* Ker $\delta_1\cong$ Ker δ_2 *as Lie algebras and also as bialgebras with trivial cobracket.*

The characteristic bi-derivation. Let $(\mathfrak{g},[-,-],\delta)$ be a Lie bialgebra. The characteristic endomorphism $\mathcal{D}:\mathfrak{g}\to\mathfrak{g}$ given by

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\delta"] \arrow[rr, bend right, "\mathcal{D}:=[-,-]\circ\delta"'] & \Lambda^2\mathfrak{g} \arrow[r, "{[-,-]}"] & \mathfrak{g} \end{tikzcd}$$

is clearly preserved by Lie bialgebra isomorphisms. Namely, if $\phi:\mathfrak{g}\to\mathfrak{g}'$ is a Lie bialgebra isomorphism and $\mathcal{D}_{\mathfrak{g}}$ and $\mathcal{D}_{\mathfrak{g}'}$ denote the endomorphism associated to $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, then $\mathcal{D}_{\mathfrak{g}'}=\phi\mathcal{D}_{\mathfrak{g}}\phi^{-1}$. As

a consecuence, any function in $\mathcal{D}$, which is invariant under conjugation, provides an invariant of the isomorphism class of the Lie bialgebra. For example, $\det(\mathcal{D})$ and $\mathrm{tr}(\mathcal{D})$ are (real) numerical invariants. The characteristic polynomial of $\mathcal{D}$ and its Jordan form are also invariants. Lie bialgebras $\mathfrak{g}$ such that $\mathcal{D}_{\mathfrak{g}} = 0$ are called **involutive**, but, in many cases $\mathcal{D}$ is far from being zero. The following proposition is standard.

Proposition 1.5. *Let $\mathfrak{g}$ be a Lie bialgebra and $\mathcal{D} = [-,-] \circ \delta$. Then $\mathcal{D}$ is a derivation with respect to the bracket and a coderivation with respect to the cobracket.*

Proof. We will prove that $\mathcal{D}$ is a derivation, the second claim follows by dualization. Let us write the 1-cocycle condition using Sweedler-type notation: $\delta[x,y] = [\delta x, y] + [x, \delta y] = [x_1 \wedge x_2, y] + [x, y_1 \wedge y_2] = [x_1, y] \wedge x_2 + x_1 \wedge [x_2, y] + [x, y_1] \wedge y_2 + y_1 \wedge [x, y_2]$. Then it follows that $\mathcal{D}[x,y] = [[x_1, y], x_2] + [x_1, [x_2, y]] + [[x, y_1], y_2] + [y_1, [x, y_2]]$. Using Jacobi identity and the definition of $\mathcal{D}$, we get

$$\mathcal{D}[x,y] = [[x_1, x_2], y] + [x, [y_1, y_2]] = [\mathcal{D}x, y] + [x, \mathcal{D}y]$$

$\square$

2 Two dimensional Lie bialgebras

In a similar way that one proves that there are only two isoclasses of Lie algebras of dimension 2, an easy manipulation of basis shows that the following list exhausts the isoclases of two dimensional Lie bialgebras. The structure is given in a basis $\{h, x\}$ of $\mathfrak{g}$.

2-dimensional Lie bialgebras isomorphism classes

$\mathfrak{g}$	$\mathfrak{g}^*$	$[-,-]$	δ	Invariants	Name	$\mathrm{tr}(\mathcal{D})$
abelian	abelian	0	0			0
abelian	non abel	0	$\delta h = x \wedge h; \delta x = 0$			0
$\mathfrak{aff}(\mathbb{R})$	abelian	$[h,x] = x$	0			0
$\mathfrak{aff}(\mathbb{R})$	non abel	$[h,x] = x$	$\delta h = h \wedge x; \delta x = 0$	Ker $\delta = [\mathfrak{g},\mathfrak{g}]$; $\delta = \partial r, r = h \wedge x$	$\mathfrak{aff}_{2,0}$	0
$\mathfrak{aff}(\mathbb{R})$	non abel	$[h,x] = x$	$\delta h = 0; \mu \neq 0$ $\delta x = \mu h \wedge x;$	Ker $\delta \neq [\mathfrak{g},\mathfrak{g}]$	$\mathfrak{aff}_{2,\mu}$	μ

where $\mathfrak{aff}(\mathbb{K})$ is the non-abelian 2-dimensional Lie algebra over $\mathbb{K}$.

The first four lines are clearly non isomorphic among them, and non isomorphic to any of the last line. Finally, thanks to the invariant given by the trace of the characteristic derivation, one sees that they are not isomorphic to each other for different μ. The same table is valid for any field $\mathbb{K}$, replacing $\mathfrak{aff}(\mathbb{R})$ by $\mathfrak{aff}(\mathbb{K})$.

Remark 2.1. *A similar table appears in [K-S], but without the parameter μ, which can not be eliminated, because $\mathrm{tr}(\mathcal{D})$ is an invariant of the Lie bialgebra.*

Simply by inspection, notice the following:

Proposition 2.2. *If $(\mathfrak{g}, [-,-], \delta)$ is a Lie bialgebra with $\dim \mathfrak{g} = 2$ and $\mathcal{D} = [-,-] \circ \delta$, then, whithin the non abelian and non co-abelian cases, $\mathrm{tr}(\mathcal{D})$ is a total invariant. With notation as in the table, $\mathfrak{aff}_{2,\mu} \cong \mathfrak{aff}_{2,\mu'}$ if and only if $\mu = \mu'$.*

Corollary 2.3. *Let $a, b, c, d \in \mathbb{R}$ such that $(a,b) \neq (0,0)$ and $(c,d) \neq (0,0)$; consider $\mathfrak{g}_{abcd}$ the Lie bialgebra given by $[h,x] = ah + bx$, $\delta h = ch \wedge x$, $\delta x = dh \wedge x$, then $\mathfrak{g}_{abcd} \cong \mathfrak{aff}_{2,\mu}(\mathbb{R})$ with $\mu = ac + bd$.*

Proof. Since $(a,b) \neq (0,0)$ and $(c,d) \neq (0,0)$ we are not in the abelian or co-abelian case. It suffices to compute the trace of $\mathcal{D}$. The computations $\mathcal{D}(h) = c[h,x]$ and $\mathcal{D}(x) = d[h,x]$ give $\mathrm{tr}(\mathcal{D}) = ac + db$. $\square$

Automorphisms groups in the non abelian and non co-abelian cases. Consider the ordered basis $\{h, x\}$ of $\mathfrak{g}$, then the Lie bialgebra automorphisms groups in the non abelian and non co-abelian cases are as follows:

• Case $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ with $[h, x] = x$ and $\delta h = h \wedge x;\ \delta x = 0$;

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \left(\begin{array}{cc} 1 & 0 \\ b & 1 \end{array} \right) :\ b \in \mathbb{R} \right\}$$

In particular, any of these maps is the exponential of a multiple of
$\mathcal{D} = [-,-] \circ \delta = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right)$.

• Case $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ with $[h, x] = x$ and $\delta_\mu h = 0$; $\delta_\mu x = \mu h \wedge x$:

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & d \end{array} \right) :\ 0 \neq d \in \mathbb{R} \right\}$$

Any of these maps with $d > 0$ is the exponential of a multiple of
$\mathcal{D}_\mu = [-,-] \circ \delta_\mu = \left(\begin{array}{cc} 0 & 0 \\ 0 & \mu \end{array} \right)$. The fact that the exponential of (a multiple of) the endomorphism $\mathcal{D}$ gives an automorphism of the Lie bialgebra is not surprising, since we already knew that $\mathcal{D}$ is a derivation and a coderivation.

3 Three dimensional real Lie algebras

Theorem 3.1. *[G-O-V] The following list exhausts the 3-dimensional solvable real Lie algebras:*

$$\begin{array}{rl} \mathbb{R}^3 : & \textit{the three dimensional abelian;} \\ \mathfrak{h}_3 : & [e_1, e_2] = e_3,\ \textit{the three dimensional Heisenberg;} \\ \mathfrak{r}_3 : & [e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3; \\ \mathfrak{r}_{3,\lambda} : & [e_1, e_2] = e_2,\ [e_1, e_3] = \lambda e_3,\ |\lambda| \leq 1; \\ \mathfrak{r}'_{3,\lambda} : & [e_1, e_2] = \lambda e_2 - e_3,\ [e_1, e_3] = e_2 + \lambda e_3,\ \lambda \geq 0. \end{array}$$

Denote $u = \frac{ih}{2}$, $v = \frac{x-y}{2}$, $w = \frac{i(x+y)}{2}$; the semisimple 3-dimensional real Lie algebras are

$$\mathfrak{sl}(2, \mathbb{R}) : [h, x] = 2x,\ [h, y] = -2y,\ [x, y] = h$$

$$\mathfrak{su}(2) : [u, v] = w,\ [v, w] = u,\ [w, u] = v.$$

Three dimensional real Lie bialgebras: general strategy. In order to classify all real three dimensional Lie bialgebras we will procede as follows:

1. Given a Lie algebra $\mathfrak{g}$, we find the general 1-cocycle $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$.

2. Determine when δ satisfies the co-Jacobi identity.

3. Study the action of $\mathrm{Aut}(\mathfrak{g}, [-,-])$ on the set of cobrackets δ.

4. Find a set of representatives, hence, the list of isomorphism classes of Lie bialgebras with underlying Lie algebra $\mathfrak{g}$.

To give a Lie bialgebra structure on the abelian Lie algebra $\mathbb{R}^3$ is the same as giving a Lie algebra structure on $(\mathbb{R}^3)^*$, so the list of all three dimensional Lie bialgebras with underlying Lie algebra $\mathbb{R}^3$ is in obvious bijection with the list of three dimensional Lie algebras.

Next, we proceed with the other cases: first $\mathfrak{h}_3$, the only 3-dimensional nilpotent and non abelian Lie algebra, secondly the solvable non nilpotent $\mathfrak{r}_3$, $\mathfrak{r}_{3,\lambda}$ and $\mathfrak{r}'_{3,\lambda}$, and finally the simple $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$.

3.1 The general co-Jacobi condition

If $\mathfrak{g}$ is any three dimensional Lie algebra, we will write the structure in terms of basis $\{x,\ y,\ h\}$ of $\mathfrak{g}$ and $\{x\wedge y,\ y\wedge h,\ h\wedge x\}$ for $\Lambda^2(\mathfrak{g})$. Write, with $a_i,\ b_i,\ c_i \in \mathbb{R}$, $i=1,2,3$, $\delta x = a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x$; $\delta y = b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x$; $\delta h = c_1 x\wedge y + c_2 y\wedge h + c_3 h\wedge x$.
For a linear map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$, the co-Jacobi condition is equivalent to the equations

$$\begin{aligned} -a_1b_2 + a_2(b_1 - c_3) + a_3c_2 &= 0,\\ b_1a_3 - b_2c_3 + b_3(-a_1 + c_2) &= 0,\\ c_1(a_3 - b_2) + c_2b_1 - c_3a_1 &= 0. \end{aligned}$$

4 Lie bialgebra structures on $\mathfrak{h}_3$

Recall that the Lie algebra $\mathfrak{h}_3$ has a basis $\{x,\ y,\ h\}$, with the relations $[h,x]=0$, $[h,y]=0$, $[x,y]=h$. We list general properties of $\mathfrak{h}_3$:

- $[\mathfrak{h}_3,\mathfrak{h}_3] = \mathbb{R}h$; $[\mathfrak{h}_3,[\mathfrak{h}_3,\mathfrak{h}_3]] = 0$.
- $\mathcal{Z}(\mathfrak{h}_3) = \mathbb{R}h$, $(\Lambda^2\mathfrak{h}_3)^{\mathfrak{h}_3} = \mathbb{R}y\wedge h \oplus \mathbb{R}h\wedge x$.
- The automorphisms group of $\mathfrak{h}_3$ is the following subgroup of $\mathrm{GL}(3,\mathbb{R})$:

$$\mathrm{Aut}(\mathfrak{h}_3) = \left\{ \phi_{\mu,\rho,\sigma,\nu,a,b} = \begin{pmatrix} \mu & \rho & 0\\ \sigma & \nu & 0\\ a & b & \lambda \end{pmatrix} :\ \mu\nu - \rho\sigma = \lambda \neq 0 \right\}$$

The 1-cocycle condition. Consider the basis $\{x\wedge y,\ y\wedge h,\ h\wedge x\}$ of $\Lambda^2(\mathfrak{h}_3)$ and write δ as in 3.1. Proposition 1.1 implies $\delta h = c_2 y\wedge h + c_3 h\wedge x$, namely $c_1 = 0$. The 1-cocycle condition for $[h,x]$ and $[h,y]$ is the content of the proof of this proposition, so it gives no further information in this case. Besides, the 1-cocycle for $[x,y]=h$ reads $\delta h = \delta[x,y] = [\delta x, y] + [x, \delta y]$, then

$$\begin{aligned} \delta h &= c_2 y\wedge h + c_3 h\wedge x\\ &= [a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x, y] + [x, b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x]\\ &= a_1 h\wedge y + b_1 x\wedge h \end{aligned}$$

so $c_2 = -a_1$ and $c_3 = -b_1$. Hence, then general 1-cocycle is

$$\begin{aligned} \delta(x) &= a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x;\\ \delta(y) &= b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x;\\ \delta(h) &= -a_1 y\wedge h - b_1 h\wedge x \end{aligned}$$

In matrix notation, $\delta = \begin{pmatrix} a_1 & b_1 & 0\\ a_2 & b_2 & -a_1\\ a_3 & b_3 & -b_1 \end{pmatrix}$.

The co-Jacobi condition (see 3.1) restricted to a 1-cocycle in $\mathfrak{h}_3$ reduces to

$$\begin{aligned} 2a_2b_1 - a_1(a_3 + b_2) &= 0,\\ b_1a_3 + b_1b_2 - 2a_1b_3 &= 0, \end{aligned}$$

which are not so easy to solve, so we use a dimensional reduction procedure, thanks to the results of section 1.

4.1 Consequence of the general result for $\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3]$

Lemma 4.1. *For $\mathfrak{g}=\mathfrak{h}_3$, the natural application of Lemma 1.3 $\mathrm{Aut}(\mathfrak{g})\to\mathrm{Aut}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$, defined by $\psi\mapsto\overline{\psi}$, is a split epimorphism.*

Proof. Consider the basis $\{x,y,h\}$, the splitting may be defined as

$$\begin{array}{rcl}\mathrm{Aut}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) & \to & \mathrm{Aut}(\mathfrak{g})\\ \phi=\begin{pmatrix}\mu & \rho\\ \sigma & \nu\end{pmatrix} & \mapsto & \widehat{\phi}=\left(\begin{array}{cc|c}\mu & \rho & 0\\ \sigma & \nu & 0\\ \hline 0 & 0 & \mu\nu-\rho\sigma\end{array}\right)\end{array}$$

□

We know $\dim(\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3])=2$ and $(\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3])$ is abelian. According to the section 2, there are only two classes of isomorphisms of 2-dimensional Lie bialgebras with abelian bracket: the co-abelian one and the one with $\overline{\delta}(\overline{x})=0$ and $\overline{\delta}(\overline{y})=\overline{x}\wedge\overline{y}$. Observe that, in virtue of the form of the Lie algebra automorphims, there is no lost of generality in assuming that the basis $\{\overline{x},\ \overline{y}\}$ is the one which allows us to write $\overline{\delta}$ in this form since any automorphism of $(\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3])$ may be lifted to an automorphism of $\mathfrak{h}_3$. Explicitly, we may assume $a_1=0$ and there are two possibilities for b_1, namely, $b_1=0$ or $b_1=1$. In matrix notation,

$$\delta_{b_1=0}=\begin{pmatrix}0 & 0 & 0\\ a_2 & b_2 & 0\\ a_3 & b_3 & 0\end{pmatrix}\quad\text{and}\quad\delta_{b_1=1}=\begin{pmatrix}0 & 1 & 0\\ a_2 & b_2 & 0\\ a_3 & b_3 & -1\end{pmatrix}.$$

Returning to the co-Jacobi condition with the assumption $a_1=0$, it is automatically satisfied in the case $b_1=0$, and it reduces to $b_2+a_3=0=2a_2$ if $b_1=1$.

Case $b_1=1$. In this case, $\delta=\begin{pmatrix}0 & 1 & 0\\ 0 & -a_3 & 0\\ a_3 & b_3 & -1\end{pmatrix}$. After conjugation by $\phi_{\mu,\nu,\rho,\sigma,a,b}$, we get

$$\delta'=\frac{1}{\mu\nu-\rho\sigma}\begin{pmatrix}\sigma & \nu & 0\\ \frac{b_3\sigma^2}{\mu\nu-\rho\sigma} & \frac{a\nu-a_3\mu\nu-b\sigma+b_3\nu\sigma+a_3\rho\sigma}{\mu\nu-\rho\sigma} & -\sigma\\ \frac{-a\nu+a_3\mu\nu+b\sigma+b_3\nu\sigma-a_3\rho\sigma}{\mu\nu-\rho\sigma} & \frac{b_3\nu^2}{\mu\nu-\rho\sigma} & -\nu\end{pmatrix}$$

If one wants to preserve the condition $a_1'=0$ then it must be $\sigma=0$, so $\delta'=\begin{pmatrix}0 & \frac{1}{\mu} & 0\\ 0 & \frac{a-a_3\mu}{\mu^2\nu} & 0\\ \frac{-a+a_3\mu}{\mu^2\nu} & \frac{b_3}{\mu^2} & -\frac{1}{\mu}\end{pmatrix}$.

The condition $b_1'=1$ forces $\mu=1$, so, with $\sigma=0$ and $\mu=1$, $\delta'=\begin{pmatrix}0 & 1 & 0\\ 0 & \frac{a-a_3}{\nu} & 0\\ \frac{-(a-a_3)}{\nu} & b_3 & -1\end{pmatrix}$.

Taking an automorphism with $a=a_3$ we get $a_3'=0$, namely δ changes into

$$\delta'=\delta_{b_3}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & b_3 & -1\end{pmatrix}$$

An automorphim preserving also $a_3'=0$ must have $a=0$, and in this case $\delta'=\delta$. Finally, the list of isoclasses of Lie bialgebras is given by the cobrackets $\{\delta_{b_3}:b_3\in\mathbb{R}\}$ given above. For each of these, the automorphism group of Lie bialgebras is

$$G=\left\{\phi_{\rho,\nu,b}=\begin{pmatrix}1 & \rho & 0\\ 0 & \nu & 0\\ 0 & b & \nu\end{pmatrix}:\nu\neq0,b,\rho\in\mathbb{R}\right\}$$

Case $b_1 = 0$: In this situation, co-Jacobi is automatically satisfied. Let $\delta = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{pmatrix}$ and let us compute $\delta' := (\phi \wedge \phi)^{-1}\delta\phi$ with $\phi = \phi_{\mu,\nu,\rho,\sigma,a,b}$ then

$$\delta' = \frac{1}{(\mu\nu - \rho\sigma)^2} \begin{pmatrix} 0 & 0 & 0 \\ a_2\mu^2 + \sigma\mu(a_3 + b_2) + b_3\sigma^2 & b_2\mu\nu + a_2\mu\rho + b_3\nu\sigma + a_3\rho\sigma & 0 \\ a_2\mu\rho + b_2\rho\sigma + b_3\nu\sigma + a_3\nu\mu & b_3\nu^2 + (a_3 + b_2)\rho\nu + a_2\rho^2 & 0 \end{pmatrix}$$

Although the matrix $\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$ does not correspond to a symmetric bilinear form, it changes according to the following rule:

$$\begin{pmatrix} a_2' & b_2' \\ a_3' & b_3' \end{pmatrix} = \frac{1}{(\mu\nu - \rho\sigma)^2} \begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix}$$

Namely, it changes as a bilinear form, divided by the square of the determinant $\mu\nu - \rho\sigma$. We know that it can be diagonalized; hence, we may assume that, up to isomorphism, $b_2 = a_3 = 0$. Also, with an automorphism with $\rho = \sigma = 0$, the coefficients change according to the rule $a_2' = a_2/\nu^2$, $a_3' = 0$, $b_2' = 0$, $b_3' = b_3/\mu^2$, so the full list of possibilities, up to isomorphism, are $a_2 = 0, \pm 1$ and $b = 0 \pm 1$. Using the automorphism with $\mu = \nu = 0$, $\rho = \sigma = 1$ we get $a_2' = b_2$ and $b_2' = a_3$, so the Lie bialgebra with cobracket with $a_2 = 1 = -b_3$ is isomorphic to the one with cobracket with $a_2 = -1 = -b_3$.

We also know that the signature is an invariant of bilinear forms and it is an invariant also in this case, because the difference between the action on bilinear forms and our case is the multiplication by the square of the determinant, which is a positive number. Similar consideration holds for the rank. We conclude that the list of isomorphism classes is given by δ_{a_2,b_3} obtained by choosing the parameters $(a_2, b_3) = (0,0),\ (1,1),$ $(-1,-1)$, $(1,-1)$, $(1,0)$, $(-1,0)$. This completes the proof of the following statement.

Theorem 4.2. *For the Lie algebra $\mathfrak{h}_3$ the exhaustive list of the isomorphism classes of Lie bialgebra structures is parametrized by the following set of cobrackets:*

$$\delta_{b_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} : b_3 \in \mathbb{R}, \quad \textit{and } \delta_{a_2,b_3} = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix}$$

with $(a_2, b_3) = (0,0),\ (1,1),\ (-1,-1),\ (1,-1),\ (1,0),\ (-1,0)$.

5 Lie bialgebra structures on $\mathfrak{r}_3$

Recall $\mathfrak{r}_3$ is the real Lie algebra with basis $\{x, y, h\}$ and Lie brackets given by

$$[h,x] = x,\ [h,y] = x + y,\ [x,y] = 0$$

Lemma 5.1. $\mathscr{Z}(\mathfrak{r}_3) = 0$ *and* $(\Lambda^2\mathfrak{r}_3)^{\mathfrak{r}_3} = 0$.

Proof. $\mathrm{ad}_x(ax + by + ch) = -cx = 0$ implies $c = 0$ and $0 = \mathrm{ad}_h(ax + by) = ax + b(x + y)$ implies $a = 0 = b$. If $\omega = ax \wedge y + bx \wedge h + cy \wedge h \in (\Lambda^2\mathfrak{r}_3)^{\mathfrak{r}_3}$ then

$$\mathrm{ad}_h(\omega) = 2ax \wedge y + (b + c)x \wedge h + cy \wedge h = 0$$

implies $a = b = c = 0$. $\square$

1-cocycles on $\mathfrak{r}_3$. Let us write, as in 3.1, with $a_i,\ b_i,\ c_i \in \mathbb{R}$, $\delta(x) = a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x$, $\delta(y) = b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x$, $\delta(h) = c_1 x\wedge y + c_2 y\wedge h + c_3 h\wedge x$. The 1-cocycle condition for $[x,y]=0$ and $\delta[x,y] = [\delta x, y] + [x, \delta y]$ give

$$\begin{aligned} 0 &= [a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x, y] + [x, b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x] \\ &= a_2 y\wedge x + a_3 y\wedge x - b_2 y\wedge x \end{aligned}$$

so $a_2 + a_3 - b_2 = 0$. Now, $[h,x] = x$ and $\delta[h,x] = [\delta h, x] + [h, \delta x]$ imply

$$\begin{aligned} & a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x \\ =\ & [c_1 x\wedge y + c_2 y\wedge h + c_3 h\wedge x, x] + [h, a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x] \\ =\ & c_2 y\wedge x + 2a_1 x\wedge y + a_2 (x+y)\wedge h + a_3 h\wedge x \end{aligned}$$

so $a_1 = -c_2 + 2a_1$ and $a_3 = -a_2 + a_3$. This is equivalent to $a_1 = c_2$ and $a_2 = 0$. Finally, $[h,y] = x + y$ and $\delta[h,y] = [\delta h, y] + [h, \delta y]$ imply

$$\begin{aligned} & (a_1+b_1) x\wedge y + (a_2+b_2) y\wedge h + (a_3+b_3) h\wedge x \\ =\ & [c_1 x\wedge y + c_2 y\wedge h + c_3 h\wedge x, y] + [h, b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x] \\ =\ & c_2 y\wedge x + c_3 y\wedge x + 2b_1 x\wedge y + b_2 (x+y)\wedge h + b_3 h\wedge x \end{aligned}$$

So, $a_1 + b_1 = -c_2 - c_3 + 2b_1$, $a_2 + b_2 = b_2$, $a_3 + b_3 = -b_2 + b_3$. Solving all the linear equations, we obtain $a_2 = a_3 = b_2 = 0$, $c_2 = a_1$, $c_3 = b_1 - 2a_1$. Hence, a general 1-cocycle δ, is given by

$$\begin{aligned} \delta(x) &= a_1 x\wedge y \\ \delta(y) &= b_1 x\wedge y \qquad\qquad\quad + b_3 h\wedge x \\ \delta(h) &= c_1 x\wedge y + a_1 y\wedge h \quad + (b_1 - 2a_1) h\wedge x \end{aligned}$$

Or, in matrix notation:

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & a_1 \\ 0 & b_3 & b_1 - 2a_1 \end{pmatrix}$$

The co-Jacobi condition for a general 1-cocycle is simply $2a_1^2 = 0$. Hence, a 1-cocycle satisfying also co-Jacobi is of the form

$$\delta = \begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & b_3 & b_1 \end{pmatrix}$$

The Lie algebras automorphism group of $\mathfrak{r}_3$ is the following subgroup of $\mathrm{GL}(3,\mathbb{R})$

$$\left\{ \phi_{\mu,\rho,a,b} = \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \rho, a, b \in \mathbb{R},\ \mu \neq 0 \right\}$$

Under the action of the automorphism group, a 1-cocycle δ maps into $\delta' = \begin{pmatrix} 0 & \frac{b_1 + b b_3}{\mu} & \frac{2bb_1 + b^2 b_3 + c_1}{\mu^2} \\ 0 & 0 & 0 \\ 0 & b_3 & \frac{b_1 + bb_3}{\mu} \end{pmatrix}$, then b_3 is an invariant.

Case $b_3 \neq 0$. Taking $b = -b_1/b_3$ we get $b_1' = 0$, so we may assume $b_1 = 0$. The conditions $b_1' = 0$ is preserved only if $b = 0$, and in that case δ changes into

$$\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2} \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix}$$

So, c_1 can be chosen up to positive scalar and we can take the numbers $c = 0, \pm 1$ as representatives. We conclude that the isomorphism classes of Lie bialgebras with coefficient $b_3 \neq 0$ consist of three 1-parameter families with cobrackets:

$$\delta_{b_3,c_1} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} : c_1 = 0, 1, -1,\ b_3 \neq 0$$

For $c_1 = 0$, the automorphism group consists of $\left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu, a \in \mathbb{R}, \mu \neq 0 \right\}$, and for $c_1 = \pm 1$, we have $c_1' = c_3/\mu^2$, so μ^2 must be equal to 1, and the automorphism group is $\left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R}, \mu = \pm 1 \right\}$.

Case $b_3 = 0$. We have then $\delta = \begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} 0 & b_1/\mu & (2bb_1 + c_1)/\mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & b_1/\mu \end{pmatrix}$ hence, b_1 is determined up to a multiple; we consider the cases $b_1 \neq 0$ and $b_1 = 0$.

Case $b_1 \neq 0$. We may assume $b_1 = 1$ then $\delta' = \begin{pmatrix} 0 & 1/\mu & (2b + c_1)/\mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\mu \end{pmatrix}$. If we whish to preserve $b_1 = 1$ we need to impose that $\mu = 1$; we obtain $\delta' = \begin{pmatrix} 0 & 1 & 2b + c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We may choose $b = -\frac{c_1}{2}$, so the new $c_1' = 0$ and take as a representative of the class of isomorphism $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The automorphisms group of this Lie bialgebra is

$$G_{b_1 \neq 0} = \left\{ \begin{pmatrix} 1 & \rho & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \rho, a \in \mathbb{R}, \right\}$$

Case $b_1 = 0$. We have then

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} 0 & 0 & c_1/\mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, we obtain that $c_1 = 0, \pm 1$ are all the possibilities for c_1. The automorphisms group of such Lie bialgebra with $c_1 = 0$ is

$$G_{b_1 = c_1 = 0} = \left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \rho, a, b \in \mathbb{R}, \mu \neq 0 \right\}$$

On the other hand, the automorphisms group of the Lie bialgebras classes with $c_1 = \pm 1$ consists of the Lie algebra maps satisfying $\mu^2 = 1$; hence

$$G_{b_1 = 0, c_1 \neq 0} = \left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \rho, a, b \in \mathbb{R}, \mu = \pm 1 \right\}$$

Theorem 5.2. *The isomorphism classes of Lie bialgebra structures on $\mathfrak{r}_3$ is given by the following list of cobrackets:*

$$\delta_{b_1 \neq 0, b_3 = 0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and } \delta_{c_1, b_3} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} : b_3 \in \mathbb{R},\ c_1 = 0, \pm 1$$

6 Lie bialgebra structures on $\mathfrak{r}_{3,\lambda}$, with $|\lambda| \leq 1$.

Recall that $\mathfrak{r}_{3,\lambda}$ is the Lie algebra with bases $\{x,\ y,\ h\}$, and bracket $[h,x]=x$, $[h,y]=\lambda y$, $[x,y]=0$. We list general properties for $\mathfrak{r}_{3,\lambda}$:

- if $\lambda \neq 0$ then $\mathcal{Z}(\mathfrak{g})=0$; if $\lambda=0$ then $\mathcal{Z}(\mathfrak{g})=\langle y\rangle$.
- If $\lambda=-1$ then $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}=\langle x\wedge y\rangle$. If $\lambda\neq -1$ then $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}=0$.

Proposition 6.1. *All the 1-cocycles on the Lie algebra $\mathfrak{r}_{3,\lambda}$ with $|\lambda|\leq 1$ are*

$$\delta=\begin{pmatrix} a_1 & \lambda c_3 & c_1 \\ 0 & \lambda a_3 & \lambda a_1 \\ a_3 & 0 & c_3 \end{pmatrix} \text{ if } \lambda\neq 1, \qquad \delta=\begin{pmatrix} a_1 & c_3 & c_1 \\ a_2 & a_3 & a_1 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ if } \lambda=1$$

where we consider the basis $\{x\wedge y,\ y\wedge h,\ h\wedge x\}$ of $\Lambda^2(\mathfrak{g})$ and notations as in 3.1.

Proof. Let $\delta:\mathfrak{g}\to\Lambda^2(\mathfrak{g})$ be a 1-cocyle, then $\delta[h,x]=[\delta h,x]+[h,\delta x]$ and $[h,x]=x$ imply

$$\begin{aligned} & a_1 x\wedge y+a_2 y\wedge h+a_3 h\wedge x \\ = \ & [c_1 x\wedge y+c_2 y\wedge h+c_3 h\wedge x, x]+[h, a_1 x\wedge y+a_2 y\wedge h+a_3 h\wedge x] \\ = \ & c_2 y\wedge x+a_1(1+\lambda)x\wedge y+\lambda a_2 y\wedge h+a_3 h\wedge x \end{aligned}$$

We conclude $\lambda a_1=c_2$, $a_2=\lambda a_2$, so $a_2=0$ if $\lambda\neq 1$, and no condition in a_2 for $\lambda=1$. In an analogous way, using the cocycle condition $\delta[h,y]=[\delta h,y]+[h,\delta y]$ for $[h,y]=\lambda y$, we get

$$\begin{aligned} & \lambda(b_1 x\wedge y+b_2 y\wedge h+b_3 h\wedge x) \\ & =[c_1 x\wedge y+c_2 y\wedge h+c_3 h\wedge x, y]+[h, b_1 x\wedge y+b_2 y\wedge h+b_3 h\wedge x] \\ & =\lambda c_3 y\wedge x+(1+\lambda)b_1 x\wedge y+\lambda b_2 y\wedge h+b_3 h\wedge x \end{aligned}$$

so $b_1=\lambda c_3$ and $b_3=\lambda b_3$, so again $b_3=0$ if $\lambda\neq 1$ and no restriction on b_3 for $\lambda=1$. The third condition is $\delta[x,y]=[\delta x,y]+[x,\delta y]$; since $[x,y]=0$, we get

$$0=[a_1 x\wedge y+a_2 y\wedge h+a_3 h\wedge x, y]+[x, b_1 x\wedge y+b_2 y\wedge h+b_3 h\wedge x]=\lambda a_3 y\wedge x-b_2 y\wedge x$$

so $b_2=\lambda a_3$. As a consequence, the general form of a 1-cocycle is, for $\lambda\neq 1$:

$$\begin{aligned} \delta(x) &= a_1 x\wedge y && && + \ a_3 h\wedge x \\ \delta(y) &= \lambda c_3 x\wedge y && + \ \lambda a_3 y\wedge h && \\ \delta(h) &= c_1 x\wedge y && + \ \lambda a_1 y\wedge h && + \ c_3 h\wedge x \end{aligned}$$

and for $\lambda=1$:

$$\begin{aligned} \delta(x) &= a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x \\ \delta(y) &= c_3 x\wedge y + a_3 y\wedge h + b_3 h\wedge x \\ \delta(h) &= c_1 x\wedge y + a_1 y\wedge h + c_3 h\wedge x \end{aligned}$$

□

Case $\mathfrak{r}_{3,\lambda}$ with $|\lambda|\leq 1, \lambda\neq\pm 1$.

Proposition 6.2. *The automorphism group of the Lie algebra $\mathfrak{r}_{3,\lambda}$ with $\lambda\neq\pm 1$, is the following subgroup of* $\mathrm{GL}(3,\mathbb{R})$*:*

$$\mathrm{Aut}(\mathfrak{g})=\left\{\begin{pmatrix} \mu & 0 & a \\ 0 & \nu & b \\ 0 & 0 & 1 \end{pmatrix} : \ \mu\nu\neq 0\right\}$$

Proof. Let us see that if $\phi : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of Lie algebra, then

$$\phi(x) = \mu x;\ \phi(y) = \nu y;\ \phi(h) = h + ax + by$$

for some $\mu, \nu, a, b \in \mathbb{R},\ \mu, \nu \neq 0$. Actually, the elements x and y may be characterized, up to scalar multiple, as the generators of $[\mathfrak{g}, \mathfrak{g}]$ and eigenvectors of ad_z, for all $z \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]$. Moreover, given such z, the element x distinguishes from y as being the eigenvector corresponding to the eigenvalue with smaller absolute value. Explicitely, if $z \notin [\mathfrak{g}, \mathfrak{g}]$, $z = ch + ax + by$ with $c \neq 0$,

$$\mathrm{ad}_z(x) = [ch + ax + by, x] = cx, \quad \mathrm{ad}_z(y) = [ch + ax + by, y] = c\lambda y$$

where $|\lambda| < 1$. This implies $\phi(x) = \mu x$ and $\phi(y) = \nu y$, for some $\mu,\ \nu \neq 0$.

Let $\phi(x) = \tilde{x} = \mu x$, $\phi(y) = \tilde{y} = \nu y$, $\phi(h) = \tilde{h} = ch + ax + by$; if ϕ is a Lie algebra morphism, then $[\tilde{h}, \tilde{x}] = \tilde{x}$, so $c = 1$. □

Let ϕ be as before, and let $\delta' := (\phi \wedge \phi)^{-1}\delta\phi$; explicitly

$$\delta' = \begin{pmatrix} \frac{a_1 + a_3 b}{\nu} & \lambda\frac{c_3 + a a_3}{\mu} & \frac{c_1 + a(a_1 + a_3 b)(1+\lambda) + b c_3(1+\lambda)}{\mu\nu} \\ 0 & \lambda a_3 & \lambda\frac{a_1 + a_3 b}{\nu} \\ a_3 & 0 & \frac{c_3 + a a_3}{\mu} \end{pmatrix}$$

Notice that, if $a_3 \neq 0$, by means of an automorphism with $b = -a_1/a_3$ and $a = -c_3/a_3$, we get δ' with $a_1' = 0 = c_3'$; explicitly $\delta' = \begin{pmatrix} 0 & 0 & \frac{a_3 c_1 - a_1 c_3(1+\lambda)}{\mu\nu} \\ 0 & \lambda a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_1' \\ 0 & \lambda a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}$.

Co-Jacobi condition for δ (recall $\lambda \neq \pm 1$) is $(1-\lambda)(a_3 c_1 - a_1 c_3(1+\lambda)) = 0$, or equivalently $a_3 c_1 - a_1 c_3(1+\lambda)) = 0$. If $a_3 = 0$, this condition reduces to $a_1 b_1 = 0$. Note that, up to isomorphism, we may independently change a_1 and b_1 by $a_1' = a_1/\nu$ and $c_3' = c_3/\mu$ respectively. But also, because one of them is zero, we have the following possibilities:

- $(a_1, c_3) = (0, 0)$, $c_1 = 0$ or 1 because c_1 is determined (up to isomorphism) up to scalar multiple.
- $(a_1, c_3) = (1, 0)$, and c_1 changes into $c_1' = (c_1 + a(1+\lambda))/\mu$ (we need $\nu = 1$ in order to preserve $a_1' = 1$), we see that we can choose a such that $c_1' = 0$.
- $(a_1, c_3) = (0, 1)$, and c_1 changes into $c_1' = (c_1 + b(1+\lambda))/\nu$, so we can also choose $c_1 = 0$.

If $a_3 \neq 0$, we may assume $a_1 = 0 = c_3$, then co-Jacobi implies $c_1 = 0$. Hence, every cocycle with $a_3 \neq 0$ satisfying co-Jacobi is equivalent to

$$\left\{ \delta_{a_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_3\lambda & 0 \\ a_3 & 0 & 0 \end{pmatrix} : 0 \neq a_3 \in \mathbb{R} \right\}$$

The parameter a_3 can not be modified using a Lie algebra automorphism, it is an invariant; we have a 1-parameter family of isomorphism classes, parametrized by a_3.

The bialgebra automorphisms are of the form $\phi = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu,\ \nu \neq 0$. We have finished the proof of the next statement.

Theorem 6.3. *The set of representatives of all isomorphisms classes of Lie bialgebras on* $\mathfrak{r}_{3,\lambda} : \lambda \neq \pm 1$ *is given by the following cobrackets*

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} : a_3 \in \mathbb{R} \right\}$$

Case $\mathfrak{g} = \mathfrak{r}_{3,\lambda}$ *with* $\lambda = -1$. Let us recall that $\mathfrak{r}_{3,\lambda=-1}$ is the Lie algebra with bases $\{x,\ y,\ h\}$, and bracket $[h,x] = x$, $[h,y] = -y$, $[x,y] = 0$. The automorphisms group of $\mathfrak{r}_{3,\lambda=-1}$, in the ordered basis $\{x,\ y,\ h\}$, identifies with the following subgroup of $\mathrm{GL}(3,\mathbb{R})$:

$$\mathrm{Aut}(\mathfrak{r}_{3,\lambda=-1}) = \left\langle \phi_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ \phi_{\mu,\nu,a,b} = \begin{pmatrix} \mu & 0 & a \\ 0 & \nu & b \\ 0 & 0 & 1 \end{pmatrix} :\ \mu\nu \neq 0 \right\rangle$$

In fact, $\phi_{\mu,\nu,a,b}$ is an automorphism by the same reasons as above, but in this case, the absolute value of the eigenvalues of x and y are the same. Actually,

$$x \mapsto y; \quad y \mapsto x; \quad h \mapsto -h$$

is an automorphism, which we denote by ϕ_0. Moreover, any automorphism is obtained by compositions of the above ones.

The set of 1-cocycles was computed for any λ but for convenience in case $\lambda = -1$ we parametrize them by $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & -a_3 & -a_1 \\ a_3 & 0 & -b_1 \end{pmatrix}$ instead of $\delta = \begin{pmatrix} a_1 & \lambda c_3 & c_1 \\ 0 & \lambda a_3 & \lambda a_1 \\ a_3 & 0 & c_3 \end{pmatrix}$. For these cocycles, co-Jacobi reads $2a_3c_1 = 0$. The action of the automorphism group is

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & -a_3 & -a_1 \\ a_3 & 0 & -b_1 \end{pmatrix} \xmapsto{\phi_{\mu,\nu,a,b}} \delta' = \begin{pmatrix} \frac{a_1+a_3b}{\nu} & \frac{b_1-aa_3}{\mu} & \frac{c_1}{\mu\nu} \\ 0 & -a_3 & \frac{-a_1-a_3b}{\nu} \\ a_3 & 0 & \frac{-b_1+aa_3}{\mu} \end{pmatrix}$$

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & -a_3 & -a_1 \\ a_3 & 0 & -b_1 \end{pmatrix} \xmapsto{\phi_0} \delta' = \begin{pmatrix} -b_1 & -a_1 & c_1 \\ 0 & a_3 & b_1 \\ -a_3 & 0 & -b_1 \end{pmatrix}$$

Case $a_3 \neq 0$*:* Co-Jacobi implies $c_1 = 0$. But also, taking an automorphism ϕ with parameters $a = b_1/a_3$ and $b = -a_1/a_3$ we get δ' with $a_1' = b_1' = 0$. Besides, using ϕ_0, $a_3 \mapsto a_3' = -a_3$, so we may choose $a_3 > 0$. We conclude that inside this isomorphism class, we have the representative

$$\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} : a_3 > 0$$

Case $a_3 = 0$*:* In this case, co-Jacobi condition is automatically satisfied. For each 3-uple (a_1, b_1, c_1) we have $\delta_{a_1,b_1,c_1} \cong \delta_{\frac{a_1}{\mu},\frac{b_1}{\nu},\frac{c_1}{\mu\nu}}$ *i.e.*

$$\delta_{a_1,b_1,c_1} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & -a_1 \\ 0 & 0 & -b_1 \end{pmatrix} \cong \delta_{\frac{a_1}{\mu},\frac{b_1}{\nu},\frac{c_1}{\mu\nu}} = \begin{pmatrix} \frac{a_1}{\nu} & \frac{b_1}{\mu} & \frac{c_1}{\mu\nu} \\ 0 & 0 & \frac{-a_1}{\nu} \\ 0 & 0 & \frac{-b_1}{\mu} \end{pmatrix}$$

By means of the isomorphism ϕ_0 we obtain aditionally $\delta_{a_1,b_1,c_1} \cong \delta_{-b_1,-a_1,c_1}$. Choosing conveniently μ and ν, we arrive at the following list of iso classes:

$$\delta_{0,0,0} = 0; \quad \delta_{0,0,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \delta_{1,0,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\delta_{1,0,1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}; \quad \delta_{1,1,c_1} = \begin{pmatrix} 1 & 1 & c_1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

where we identify $\delta_{1,0,0} \cong \delta_{0,1,0}$; $\delta_{1,0,1} \cong \delta_{0,1,1}$.

7 Lie bialgebra structures on $\mathfrak{r}_{3,\lambda}$ with $\lambda = 1$.

Recall that $\mathfrak{r}_{3,1}$ is the Lie algebra with ordered bases $\{x,\ y,\ h\}$ and bracket determined by $[h,x] = x$, $[h,y] = y$, $[x,y] = 0$. It can be easily verified that the automorphism group is the subgroup of $\mathrm{GL}(3,\mathbb{R})$ expressed as matrices as:

$$\mathrm{Aut}(\mathfrak{r}_{3,1}) = \left\{ \phi^{a,b}_{\mu,\nu,\rho,\sigma} = \begin{pmatrix} \mu & \rho & a \\ \sigma & \nu & b \\ 0 & 0 & 1 \end{pmatrix} :\ \mu\nu - \rho\sigma \neq 0 \right\}$$

Recall that the 1-cocycles are in matrix notation given by: $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & a_3 & a_1 \\ a_3 & b_3 & b_1 \end{pmatrix}$ with $a_1, a_2, a_3, b_1, b_3, c_1 \in \mathbb{R}$.

The co-Jacobi identity is always satisfied.

Theorem 7.1. *For the Lie algebra $\mathfrak{g} = \mathfrak{r}_{3,\lambda=1}$ the exhaustive list of representatives of the isomorphism classes of Lie bialgebras, or equivalently, the exhaustive list of a representative set of cobrackets is the following :*

$$\begin{tikzcd}
 & a_2 \neq 0 \arrow[r, Rightarrow] & \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad : a_2 > 0, c_1 = 0, \pm 1 & \\
b_3 \neq 0 \arrow[ur] \arrow[dr] & & a_1 \neq 0 \arrow[r, Rightarrow] & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
 & a_2 = 0 \arrow[ur] \arrow[r] & a_1 = 0 \arrow[r, Rightarrow] & \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : c_1 = 0, \pm 1
\end{tikzcd}$$

$$\begin{tikzcd}
 & a_3 \neq 0 \arrow[r, Rightarrow] & \begin{pmatrix} 0 & 0 & c_1 \\ 0 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} \quad : a_3 > 0, c_1 = 0, 1 & & \\
b_3 = 0 \arrow[ur] \arrow[dr] & & & b_1 = 0 \arrow[r, Rightarrow] & \begin{pmatrix} 0 & 0 & c_1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0, \pm 1 \\
 & a_3 = 0 \arrow[r] \arrow[dr] & a_2 \neq 0 \arrow[ur] \arrow[r] & b_1 \neq 0 \arrow[r, Rightarrow] & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 & & a_2 = 0 \arrow[r] \arrow[dr] & (a_1, b_1) \neq (0,0) \arrow[r, Rightarrow] & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
 & & & (a_1, b_1) = (0,0) \arrow[r, Rightarrow] & \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0, 1
\end{tikzcd}$$

We dedicate the rest of the section to the proof of this result, which proceeds in the cases given by the previous diagram.

Action of the automorphism group. $\delta' = (\phi \wedge \phi)^{-1} \delta \phi$, $\phi = \phi^{a,b}_{\mu,\nu,\rho,\sigma}$, then $\delta' =$

$$\begin{pmatrix} \frac{\mu(a_1+aa_2+a_3b)+\sigma(aa_3+b_1+bb_3)}{\mu\nu-\rho\sigma} & \frac{\nu(aa_3+b_1+bb_3)+\rho(a_1+aa_2+a_3b)}{\mu\nu-\rho\sigma} & \frac{a(aa_2+2a_1+2a_3b)+2bb_1+b^2b_3+c_1}{\mu\nu-\rho\sigma} \\ \frac{a_2\mu^2+\sigma(2a_3\mu+b_3\sigma)}{\mu\nu-\rho\sigma} & \frac{a_3\mu\nu+a_2\mu\rho+b_3\nu\sigma+a_3\rho\sigma}{\mu\nu-\rho\sigma} & \frac{\mu(a_1+aa_2+a_3b)+\sigma(aa_3+b_1+bb_3)}{\mu\nu-\rho\sigma} \\ \frac{a_3\mu\nu+a_2\mu\rho+b_3\nu\sigma+a_3\rho\sigma}{\mu\nu-\rho\sigma} & \frac{b_3\nu^2+\rho(2a_3\nu+a_2\rho)}{\mu\nu-\rho\sigma} & \frac{\nu(aa_3+b_1+bb_3)+\rho(a_1+aa_2+a_3b)}{\mu\nu-\rho\sigma} \end{pmatrix}$$

Considering the special type of automorphisms with $\rho = 0 = \sigma$, we obtain

$$\delta' = \begin{pmatrix} \frac{a_1+aa_2+a_3b}{\nu} & \frac{aa_3+b_1+bb_3}{\mu} & \frac{a^2a_2+a(2a_1+2a_3b)+2bb_1+b^2b_3+c_1}{\mu\nu} \\ \frac{a_2\mu}{\nu} & a_3 & \frac{a_1+aa_2+a_3b}{\nu} \\ a_3 & \frac{b_3\nu}{\mu} & \frac{aa_3+b_1+bb_3}{\mu} \end{pmatrix}$$

If also $\mu = \nu = 1$ and $a = 0$, we get

$$\delta' = \begin{pmatrix} a_1+a_3b & b_1+bb_3 & 2bb_1+b^2b_3+c_1 \\ a_2 & a_3 & a_1+a_3b \\ a_3 & b_3 & b_1+bb_3 \end{pmatrix}$$

Case $b_3 \neq 0$. From the previous analysis, we see that if $b_3 \neq 0$, we can choose b such that b_1' transforms into zero. Explicitly, take $b = -b_1/b_3$. So we may assume from the beginning, that $b_1 = 0$. Consider now δ of type $\delta = \begin{pmatrix} a_1 & 0 & c_1 \\ a_2 & a_3 & a_1 \\ a_3 & b_3 & 0 \end{pmatrix}$. If we use an automorphism with $\rho = 0$ we get $\delta' = \begin{pmatrix} \frac{\mu(a_1+aa_2+a_3b)+\sigma(aa_3+bb_3)}{\mu\nu} & \frac{aa_3+bb_3}{\mu} & * \\ * & * & * \\ * & * & * \end{pmatrix}$. Then we can preserve the condition $b_1' = 0$ if we set for example $a = b_3$, $b = -a_3$. In this case,

$$\delta' = \frac{1}{\mu\nu}\begin{pmatrix} (a_1-a_3^2+a_2b_3)\mu & 0 & 2a_1b_3-a_3^2b_3+a_2b_3^2+c_1 \\ a_2\mu^2+\sigma(2a_3\mu+b_3\sigma) & \nu(a_3\mu+b_3\sigma) & (a_1-a_3^2+a_2b_3)\mu \\ \nu(a_3\mu+b_3\sigma) & b_3\nu^2 & 0 \end{pmatrix}$$

Since $b_3 \neq 0$, we may choose $\sigma = -a_3\mu/b_3$ to get δ' with $a_3' = 0$, so we may assume from the beginning that $a_3 = 0$, *i.e.* $\delta = \begin{pmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & a_1 \\ 0 & b_3 & 0 \end{pmatrix}$. Using an automorphism with $b = \rho = \sigma = 0$, $\nu = 1$, we get $\delta' = \begin{pmatrix} a_1+aa_2 & 0 & \frac{2aa_1+a^2a_2+c_1}{\mu} \\ a_2\mu & 0 & a_1+a_2 \\ 0 & \frac{b_3}{\mu} & 0 \end{pmatrix}$. Notice that we can make $b_3' = 1$ by means of $\mu = b_3$ so, we may assume $b_3 = 1$. We distinguish two subcases: $a_2 \neq 0$ and $a_2 = 0$.
If $a_2 \neq 0$, we can choose $a = -a_1/a_2$ and get $a_1' = 0$, so we start from the beginning with

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Now, with a general automorphism,

$$\delta' = \frac{1}{\mu\nu-\rho\sigma}\begin{pmatrix} aa_2\mu+b\sigma & b\nu+aa_2\rho & a^2a_2+c_1 \\ a_2\mu^2+\sigma^2 & a_2\mu\rho+\nu\sigma & aa_2\mu+b\sigma \\ a_2\mu\rho+\nu\sigma & \nu^2+a_2\rho^2 & b\nu+aa_2\rho \end{pmatrix}$$

In order to preserve the conditions $a_1' = b_1' = a_3' = 0$ we must solve the equations

$$aa_2\mu + b\sigma = 0 \tag{1}$$
$$b\nu + aa_2\rho = 0 \tag{2}$$
$$a_2\mu\rho + \nu\sigma = 0 \tag{3}$$

Computing $(1)\nu-(2)\sigma$ we get $aa_2(\mu\nu-\rho\sigma)=0$. Since $a_2\neq 0\neq \mu\nu-\rho\sigma$, we conclude $a=0$. Going back to δ' but with $a=0$ we have

$$\delta' = \frac{1}{\mu\nu-\rho\sigma}\begin{pmatrix} b\sigma & b\nu & c_1 \\ a_2\mu^2+\sigma^2 & a_2\mu\rho+\nu\sigma & b\sigma \\ a_2\mu\rho+\nu\sigma & \nu^2+a_2\rho^2 & b\nu \end{pmatrix}$$

Since ν and σ can not be simultaneously zero, it must be $b=0$, explicitly

$$\delta' = \frac{1}{\mu\nu-\rho\sigma}\begin{pmatrix} 0 & 0 & c_1 \\ a_2\mu^2+\sigma^2 & a_2\mu\rho+\nu\sigma & 0 \\ a_2\mu\rho+\nu\sigma & \nu^2+a_2\rho^2 & 0 \end{pmatrix}$$

If $a_2<0$ then there exists an automorphism with convenient μ and σ such that $a_2'=0$, but this case will be considered later, so assume $a_2>0$. The condition $a_3'=0$ means $a_2\mu\rho+\nu\sigma=0$. If $\mu\neq 0$, we can solve $\rho=-\nu\sigma/(a_2\mu)$ and get

$$\delta' = \begin{pmatrix} 0 & 0 & \frac{a_2c_1\mu}{\nu(a_2\mu^2+\sigma^2)} \\ \frac{a_2\mu}{\nu} & 0 & 0 \\ 0 & \frac{\nu}{\mu} & 0 \end{pmatrix}$$

In order to get $b_3'=1$ we need $\mu=\nu$, so $\delta' = \begin{pmatrix} 0 & 0 & \frac{a_2c_1}{a_2\mu^2+\sigma^2} \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

It is clear that applying automorphisms with $\mu\neq 0$, a_2 is an invariant and c_1 may be chosen up to positive scalar. If $\mu=0$, we get $\delta'=\frac{1}{-\rho\sigma}\begin{pmatrix} 0 & 0 & c_1 \\ \sigma^2 & \nu\sigma & 0 \\ \nu\sigma & \nu^2+a_2\rho^2 & 0 \end{pmatrix}$.

If we want to preserve $a_2'>0$ and $b_2'=0$ we need $\sigma\neq 0$ and $\nu=0$, then $\delta' = \begin{pmatrix} 0 & 0 & \frac{-c_1}{\rho\sigma} \\ \frac{-\sigma}{\rho} & 0 & 0 \\ 0 & \frac{-a_2\rho}{\sigma} & 0 \end{pmatrix}$. If $b_3'=1$ then $\sigma=-a_2\rho$, hence $\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{a_2\rho^2} \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. So, a_2 is an invariant in the isomorphism class and c_1 may be chosen up to positive scalar. Hence, the list of isomorphisms classes in case $b_3\neq 0$, $a_2\neq 0$ consists of

$$\delta_{a_2,c_1} = \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : a_2>0; c_1=0,\pm 1$$

Case $a_2=0$, $\delta = \begin{pmatrix} a_1 & 0 & c_1 \\ 0 & 0 & a_1 \\ 0 & b_3 & 0 \end{pmatrix}$. Applying $\phi^{a,b}_{\mu,\nu,\rho,\sigma}$ with $a=0=b=\rho=\sigma$, $\nu=1$

$$\delta' = \begin{pmatrix} a_1 & 0 & c_1/\mu \\ 0 & 0 & a_1 \\ 0 & b_3/\mu & 0 \end{pmatrix}$$

so we may assume $b_3=1$. Using a general automorphism we obtain

$$\delta' = \frac{1}{\mu\nu-\rho\sigma}\begin{pmatrix} a_1\mu+b\sigma & b\nu+a_1\rho & 2aa_1+b^2+c_1 \\ \sigma^2 & \nu\sigma & a_1\mu+b\sigma \\ \nu\sigma & \nu^2 & b\nu+a_1\rho \end{pmatrix}$$

To preserve $a_2' = 0$ we need $\sigma = 0$, so $\delta' = \frac{1}{\mu\nu}\begin{pmatrix} a_1\mu & b\nu + a_1\rho & 2aa_1 + b^2 + c_1 \\ 0 & 0 & a_1\mu \\ 0 & \nu^2 & b\nu + a_1\rho \end{pmatrix}$. Requiring also $b_1' = 0$, then $b = -\frac{a_1\rho}{\nu}$, $\delta' = \frac{1}{\mu\nu}\begin{pmatrix} a_1\mu & 0 & 2aa_1 + \left(\frac{a_1\rho}{\nu}\right)^2 + c_1 \\ 0 & 0 & a_1\mu \\ 0 & \nu^2 & 0 \end{pmatrix}$. The condition $b_3' = 1$ implies $\mu = \nu$, then $\delta' = \begin{pmatrix} \frac{a_1}{\mu} & 0 & \frac{2aa_1 + \left(\frac{a_1\rho}{\mu}\right)^2 + c_1}{\mu^2} \\ 0 & 0 & \frac{a_1}{\mu} \\ 0 & 1 & 0 \end{pmatrix}$.

We distinguish the cases $a_1 \neq 0$ and $a_1 = 0$. If $a_1 \neq 0$, set $\mu = a_1$ to get $a_1' = 1$, then

$$\delta' = \begin{pmatrix} 1 & 0 & \left(2a + \rho^2 + c_1\right) \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then a may be chosen such that $c_1' = 0$, so the isoclass has only one representative $\delta_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. In case $a_1 = 0$ we have $\delta' = \begin{pmatrix} 0 & 0 & c_1/\mu^2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, so c_1 may be chosen up to positive scalar. Hence the set of isoclasses with $a_2 = 0$ and $b_3 \neq 0$ is

$$\left\{ \delta_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};\ \delta_{c_1} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : c_1 = 0, \pm 1 \right\}$$

Case $b_3 = 0$ If we take an automorphism $\phi_{\rho=0}$, $\delta' = \begin{pmatrix} * & \frac{aa_3 + b_1}{\mu} & * \\ \frac{a_2\mu + 2a_3\sigma}{\nu} & a_3 & * \\ a_3 & 0 & \frac{aa_3 + b_1}{\mu} \end{pmatrix}$.

Subcase $b_3 = 0$ *and* $a_3 \neq 0$. If $a_3 \neq 0$, we may choose σ and μ such that $a_2' = 0$ and a such that $b_1' = 0$, so we may start with

$$\delta = \begin{pmatrix} a_1 & 0 & c_1 \\ 0 & a_3 & a_1 \\ a_3 & 0 & 0 \end{pmatrix}$$

By means of $\phi^{a,b}_{\mu,\nu,\rho,\sigma}$ with $\rho = 0 = \sigma = a$, we get $\delta' = \begin{pmatrix} \frac{a_1 + a_3 b}{\nu} & 0 & \frac{c_1}{\mu\nu} \\ 0 & a_3 & \frac{a_1 + a_3 b}{\nu} \\ a_3 & 0 & 0 \end{pmatrix}$.

Besides, if we take a convenient b, we get $a_1' = 0$. So we may start with $\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}$; applying $\phi^{a,b}_{\mu,\nu,\rho,\sigma}$ to the previous δ, we get

$$\delta' = \frac{a_3}{\mu\nu - \rho\sigma}\begin{pmatrix} b\mu + a\sigma & a\nu + b\rho & 2ab + c_1/a_3 \\ 2\mu\sigma & \mu\nu + \rho\sigma & b\mu + a\sigma \\ \mu\nu + \rho\sigma & 2\nu\rho & a\nu + b\rho \end{pmatrix}$$

In order to preserve the conditions $a_1 = 0 = a_2 = b_1 = b_3$, we need to solve the equations $b\mu + a\sigma = 0$, $a\nu + b\rho = 0$, $\mu\sigma = 0$, $\nu\rho = 0$. The first two equations imply $a = b = 0$ and the last two say that the submatrix $\begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix}$ has to be one the two possibilities $\begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}$ or $\begin{pmatrix} 0 & \rho \\ \sigma & 0 \end{pmatrix}$. Hence, after appyling

each of these automorphisms, the given $\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}$ transforms respectively into

$$\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu\nu} \\ 0 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} \text{ or } \delta' = \begin{pmatrix} 0 & 0 & -\frac{c_1}{\rho\sigma} \\ 0 & -a_3 & 0 \\ -a_3 & 0 & 0 \end{pmatrix}$$

then we may choose a_3 up to sign and c_1 up to scalar.

We finally conclude that an exhaustive list of isomorphism classes in the case $a_3 \neq 0$, $b_3 = 0$, consists of the Lie bialgebras with the following cobrackets

$$\delta_{a_3,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}; \quad \delta_{a_3,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} : a_3 > 0$$

Subcase $b_3 = 0$ *and* $a_3 = 0$. We start in this case with $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & a_1 \\ 0 & 0 & b_1 \end{pmatrix}$; after applying a general automorphism it transforms into $\delta' =$

$$\frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} a_1\mu + aa_2\mu + b_1\sigma & b_1\nu + a_1\rho + aa_2\rho & {}_2aa_1 + a^2a_2 +_2 bb_1 + c_1 \\ a_2\mu^2 & a_2\mu\rho & a_1\mu + aa_2\mu + b_1\sigma \\ a_2\mu\rho & a_2\rho^2 & b_1\nu + a_1\rho + aa_2\rho \end{pmatrix}$$

We distinguish the cases $a_2 \neq 0$ or $a_2 = 0$. *Suppose* $b_3 = 0$, $a_3 = 0$ *and* $a_2 \neq 0$. The maps preserving $b_3 = 0$ are those with $\rho = 0$, then

$$\delta' = \frac{1}{\mu\nu} \begin{pmatrix} a_1\mu + aa_2\mu + b_1\sigma & b_1\nu & 2aa_1 + a^2a_2 + 2bb_1 + c_1 \\ a_2\mu^2 & 0 & a_1\mu + aa_2\mu + b_1\sigma \\ 0 & 0 & b_1\nu \end{pmatrix}$$

Besides, if we take $\sigma = 0$, $a = -\frac{a_1}{a_2}$, we get $a_1' = 0$; so we may start with $\delta = \begin{pmatrix} 0 & b_1 & c_1 \\ a_2 & 0 & 0 \\ 0 & 0 & b_1 \end{pmatrix}$. Applying an automorphism with $\rho = 0$, it maps to

$$\delta' = \frac{1}{\mu\nu} \begin{pmatrix} aa_2\mu + b_1\sigma & b_1\nu & a^2a_2 + 2bb_1 + c_1 \\ a_2\mu^2 & 0 & aa_2\mu + b_1\sigma \\ 0 & 0 & b_1\nu \end{pmatrix}$$

The condition $a_1' = 0$ preserves if $a = \frac{-b_1\sigma}{a_2\mu}$, so $\delta' = \begin{pmatrix} 0 & \frac{b_1}{\mu} & \frac{2bb_1 + c_1 + \frac{b_1^2\sigma^2}{a_2\mu^2}}{\mu\nu} \\ \frac{a_2\mu}{\nu} & 0 & 0 \\ 0 & 0 & \frac{b_1}{\mu} \end{pmatrix}$; then a_2 can be chosen up to scalar. If we choose $a_2 = 1$ and also $\mu = \nu$, then $\delta' = \begin{pmatrix} 0 & b_1/\mu & \frac{2bb_1 + c_1 + \frac{b_1^2\sigma^2}{\mu^2}}{\mu^2} \\ 1 & 0 & 0 \\ 0 & 0 & b_1/\mu \end{pmatrix}$. Hence b_1 is determined up to scalar, so the possibilities are $b_1 = 0$ or $b_1 = 1$.

Subcase $b_1 = 0$: $\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then c_1 may be chosen up to positive scalar. The list of isomorphisms classes in this case is given by $\left\{ \delta = \begin{pmatrix} 0 & 0 & c_1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0, \pm 1 \right\}$. *Subcase* $b_1 = 1$. Let $\mu = 1$ then

$\delta' = \begin{pmatrix} 0 & 1 & 2b+c_1+\sigma^2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We may choose $c_1 = 0$. In this case there is a unique isoclass given by $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Suppose $b_3 = 0$, $a_3 = 0$ and $a_2 = 0$. We start with $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & a_1 \\ 0 & 0 & b_1 \end{pmatrix}$. After applying a general isomorphism it is mapped into

$$\delta' = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} a_1\mu + b_1\sigma & a_1\rho + b_1\nu & 2aa_1 + 2bb_1 + c_1 \\ 0 & 0 & a_1\mu + b_1\sigma \\ 0 & 0 & a_1\rho + b_1\nu \end{pmatrix}$$

If the pair $(a_1, b_1) \neq (0,0)$ then there exists a linear transformation $\begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix}$ of determinant 1, such that $(a_1\mu + b_1\sigma, a_1\rho + b_1\nu) = (1,0)$. This says that the cobracket δ belongs to same isoclass that one with $a_1 = 1$ and $b_1 = 0$. If we make such a choice, namely $\delta = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then it transforms under a general automorphism into

$$\delta' = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} \mu & \rho & 2a + c_1 \\ 0 & 0 & \mu \\ 0 & 0 & \rho \end{pmatrix}$$

so, if we whish to preserve $b_1' = 0$, it must be $\rho = 0$, then δ' equals $\delta' = \begin{pmatrix} \frac{1}{\nu} & 0 & \frac{2a+c_1}{\mu\nu} \\ 0 & 0 & \frac{1}{\nu} \\ 0 & 0 & 0 \end{pmatrix}$. In order to preserve also $a_1' = 1$ we need $\nu = 1$; then $\delta' = \begin{pmatrix} 1 & 0 & \frac{2a+c_1}{\mu} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and we may choose $c_1 = 0$. Hence, in this case we get only one representative given by $\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

If the pair $(a_1, b_1) = (0,0)$ then

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu\nu - \rho\sigma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We conclude that there are two representatives for this case, namely

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0 \text{ or } c_1 = 1.$$

8 Lie bialgebra structures on $\mathfrak{r}'_{3,\lambda}$

In basis $\{h, x, y\}$, the Lie algebra $\mathfrak{r}'_{3,\lambda}$ has the following brackets: $[h,x] = \lambda x - y$, $[h,y] = x + \lambda y$, $[x,y] = 0$. Remark that, in the basis $\{x, y\}$, the linear transformation ad_h has matrix $\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$. Notice the similarity

with the matrix associated to multiplication by the number $\lambda - i$. A straightforward computation shows $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$ if $\lambda \neq 0$ and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = \mathbb{R}x \wedge y$ if $\lambda = 0$, for $\mathfrak{g} = \mathfrak{r}'_{3,\lambda}$.

1-cocycle condition. Sea $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ a 1-cocycle. From $0 = \delta[x,y] = [\delta x, y] + [x, \delta y]$ we get

$$\begin{aligned} 0 &= [a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x, y] + [x, b_1 x \wedge y + b_2 y \wedge h + b_3 h \wedge x] \\ &= (a_2 + a_3\lambda - b_2\lambda + b_3) y \wedge x \end{aligned}$$

hence, $a_2 + a_3\lambda - \lambda b_2 + b_3 = 0$. The 1-cocycle condition for $[h,x] = \lambda x - y$ gives

$$\begin{aligned} \lambda\delta x - \delta y &= [\delta h, x] + [h, \delta x] \\ &= [c_1 x \wedge y + c_2 y \wedge h + c_3 h \wedge x, x] + [h, a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x] \\ &= (-\lambda c_2 + c_3 + 2\lambda a_1) x \wedge y + (a_2\lambda + a_3) y \wedge h + (-a_2 + \lambda a_3) h \wedge x) \end{aligned}$$

So

$$\begin{aligned} \lambda a_1 - b_1 &= -\lambda c_2 + c_3 + 2\lambda a_1 \\ \lambda a_2 - b_2 &= a_2\lambda + a_3 \\ \lambda a_3 - b_3 &= -a_2 + \lambda a_3 \end{aligned}$$

then $\lambda a_1 + b_1 = \lambda c_2 - c_3$, $-b_2 = a_3$, $b_3 = a_2$. Similarly, $[h,y] = x + \lambda y$ gives

$$\begin{aligned} \delta x + \lambda\delta y &= [\delta h, y] + [h, \delta y] \\ &= (c_2 + c_3\lambda) y \wedge x + 2\lambda b_1 x \wedge y + (b_2\lambda + b_3) y \wedge h + (-b_2 + \lambda b_3) h \wedge x \end{aligned}$$

then

$$a_1 + \lambda b_1 = -c_2 - \lambda c_3 + 2\lambda b_1, \quad a_2 + \lambda b_2 = \lambda b_2 + b_3, \quad a_3 + \lambda b_3 = -b_2 + \lambda b_3$$

Summarizing, we get

$$a_2 = b_3, \quad a_3 = -b_2,$$

$$\lambda b_2 - b_3 = \lambda a_3 + a_2, \quad \lambda b_1 - a_1 = c_2 + \lambda c_3, \quad b_1 + \lambda a_1 = \lambda c_2 - c_3$$

The last two equations are equivalent to

$$c_2 = \frac{a_1(\lambda^2 - 1) + 2\lambda b_1}{1 + \lambda^2}, \quad c_3 = \frac{b_1(\lambda^2 - 1) - 2\lambda a_1}{1 + \lambda^2}$$

while the first three ones are equivalent to

$$a_2 = -\lambda a_3, \quad b_2 = -a_3, \quad b_3 = -\lambda a_3$$

The general 1-cocycle, in basis $\{x, y, h\}$, $\{x \wedge y, y \wedge h, h \wedge x\}$ is given by

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ -\lambda a_3 & -a_3 & \frac{a_1(\lambda^2-1)+2\lambda b_1}{1+\lambda^2} \\ a_3 & -\lambda a_3 & \frac{b_1(\lambda^2-1)-2\lambda a_1}{1+\lambda^2} \end{pmatrix}$$

For a 1-cocycle, the co-Jacobi condition is $2\frac{(a_1^2+b_1^2)\lambda + a_3 c_1(1+\lambda^2)}{1+\lambda^2} = 0$.

Proposition 8.1. *The automorphism group of the Lie algebra $\mathfrak{r}'_{3,\lambda}$, expresed as matrices in basis $\{x, y, h\}$, is the following subgroup of* $\mathrm{GL}(3, \mathbb{R})$

$$\left\{ \begin{pmatrix} \mu & -\sigma & a \\ \sigma & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \sigma, a, b \in \mathbb{R},\ \mu^2 + \sigma^2 \neq 0 \right\}$$

Proof. Using that $[\mathfrak{g},\mathfrak{g}]$ is generated by x and y and it is invariant under automorphism, we conclude that any automorphism ϕ restricted to $[\mathfrak{g},\mathfrak{g}]$ must be of the form $\phi(x)=\mu x+\rho y$ and $\phi y=\sigma x+\nu y$, with $\mu\nu-\rho\sigma\neq 0$. Also, writing $\phi(h)=ax+by+ch$, since ϕ is an automorphism of the Lie algebra, we have

$$c[h,\phi x]=[\phi h,\phi x]=\lambda\phi x-\phi y,\quad c[h,\phi y]=[\phi h,\phi y]=\phi x+\lambda\phi y,$$

in matrix notation,

$$c\begin{pmatrix}\mu & \sigma\\ \rho & \nu\end{pmatrix}\begin{pmatrix}\lambda & -1\\ 1 & \lambda\end{pmatrix}=\begin{pmatrix}\lambda & -1\\ 1 & \lambda\end{pmatrix}\begin{pmatrix}\mu & \sigma\\ \rho & \nu\end{pmatrix}$$

Taking determinant we get $c=1$, and if $\begin{pmatrix}\mu & \sigma\\ \rho & \nu\end{pmatrix}$ commutes with the matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ then it must be of the form $\begin{pmatrix}\mu & -\sigma\\ \sigma & \mu\end{pmatrix}$. $\square$

Action of the automorphisms group on 1-cocycles. The efect of an arbitrary automorphism on a general 1-cocycle is the following $\delta'=$

$$=\begin{pmatrix}\frac{a_1\mu+b_1\sigma-a_3(\mu(a\lambda-b)+\sigma(a+b\lambda))}{\mu^2+\sigma^2} & \frac{b_1\mu-a_1\sigma-a_3(b(\lambda\mu+\sigma)+a(\mu-\lambda\sigma))}{\mu^2+\sigma^2} & c_1'\\ -a_3\lambda & -a_3 & *\\ a_3 & -a_3\lambda & *\end{pmatrix}$$

with

$$c_1'=\frac{c_1(1+\lambda^2)-\lambda(-2a(b_1+a_1\lambda)+a^2a_3(1+\lambda^2)+b(2a_1-2b_1\lambda+a_3b(1+\lambda^2)))}{(\mu^2+\sigma^2)(1+\lambda^2)}$$

Recall that co-Jacobi condition reads $2\frac{(a_1^2+b_1^2)\lambda+a_3c_1(1+\lambda^2)}{1+\lambda^2}=0$, or, equivalently $(a_1^2+b_1^2)\lambda+a_3c_1(1+\lambda^2)=0$. We will make some simplifications using the automorphism group.
Case $a_3\neq 0$. If we choose an automorphism with $\mu=1$, $\sigma=0$, then δ' has $a_1'=a_1+a_3(b-\lambda a)$ and $b_1'=b_1-(\lambda b+a)$, so we can choose a and b such that $a_1'=0=b_1'$; hence, we may suppose from the begining that $a_1=0$ and $b_1=0$. But now $a_1=b_1=0$ together with co-Jacobi imply $a_3c_1(1+\lambda^2)=0$ so $a_3c_1=0$; since $a_3\neq 0$, $c_1=0$, hence

$$\delta_{a_3,\lambda}=\begin{pmatrix}0 & 0 & 0\\ -\lambda a_3 & -a_3 & 0\\ a_3 & -\lambda a_3 & 0\end{pmatrix}$$

To compute the automorphism group, notice that $\phi_{\mu,\nu,a,b}$ transforms $\delta_{a_1=b_1=c_1=0}$ into δ' with $c_1'=-\lambda a_3\frac{a^2+b^2}{\mu^2+\nu^2}$, so in case $\lambda\neq 0$ the only possibility to preserve $c_1=0$ is $a=b=0$. Hence, the automorphism group in case $a_3\neq 0$, $\lambda\neq 0$ is

$$\left\{\begin{pmatrix}\mu & -\sigma & 0\\ \sigma & \mu & 0\\ 0 & 0 & 1\end{pmatrix}:\mu,\sigma\in\mathbb{R},\ \mu^2+\sigma^2\neq 0\right\}$$

But if $\lambda=0$, $\delta_{a_1=b_1=c_1=0}$ transforms by $\phi_{\mu,\nu,a,b}$ into

$$\delta'=\begin{pmatrix}a_3\frac{b\mu-a\sigma}{\mu^2+\sigma^2} & -a_3\frac{b\sigma+a\mu}{\mu^2+\sigma^2} & 0\\ 0 & -a_3 & -a_3\frac{b\mu-a\sigma}{\mu^2+\sigma^2}\\ a_3 & 0 & a_3\frac{b\sigma+a\mu}{\mu^2+\sigma^2}\end{pmatrix}$$

Since $\begin{pmatrix}\mu & -\sigma\\ \sigma & \mu\end{pmatrix}$ is invertible, the only way to preserve $a_1=b_1=0$ is with $a=b=0$. Hence, the automorphism group in case $a_3\neq 0$, $\lambda=0$ is the same as in case $\lambda\neq 0$.

Case $a_3 = 0$. If $\lambda \neq 0$, co-Jacobi implies $a_1 = b_1 = 0$; conjugation by $\phi_{\mu,\nu,a,b}$ gives

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2+\sigma^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so c_1 can be choosen up to positive scalar. We may take $0, \pm 1$ as representatives. In case $a_3 = 0$ but $\lambda = 0$, co-Jacobi identity gives no further information. We study the action of the automorphisms group in this case. $\delta_{a_3=0,\lambda=0}$ transforms by $\phi_{\mu,\nu,a,b}$ into

$$\delta' = \begin{pmatrix} \frac{a_1\mu+b_1\sigma}{\mu^2+\sigma^2} & \frac{-a_1\sigma+b_1\mu}{\mu^2+\sigma^2} & \frac{c_1}{\mu^2+\sigma^2} \\ 0 & 0 & -\frac{a_1\mu+b_1\sigma}{\mu^2+\sigma^2} \\ 0 & 0 & \frac{a_1\sigma-b_1\mu}{\mu^2+\sigma^2} \end{pmatrix}$$

The pair (a_1, b_1) transforms as $a_1 + ib_1 \mapsto \frac{a_1+ib_1}{\mu+i\sigma}$ in the complex plane. We know that there are two orbits: $(a_1, b_1) = (0, 0)$ wich has trivial action and it gives the same cobrackets as for $\lambda \neq 0$, and $\{(a_1, b_1) \neq (0, 0)\}$, which has free $\mathbb{C}^*$-action. For the second case, one can take $(a_1, b_1) = (1, 0)$ as a representative. We conclude that a set of representatives of Lie cobrackets in case $a_3 = 0$, $\lambda = 0$ is given by

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0, \pm 1, \ \delta_{c_1}^{(1,0)} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} : c_1 \in \mathbb{R}$$

The automorphisms group of the Lie bialgebra with $\delta_{c_1}^{(1,0)}$ is

$$\left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}. \right\}$$

Theorem 8.2. *The set of isomorphism classes of Lie bialgebra with underlying Lie algebra* $\mathfrak{r}'_{3,\lambda}$ *in the case* $\lambda \neq 0$ *is given by the following list of cobrackets:*

$$\delta_{a_3,\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda a_3 & -a_3 & 0 \\ a_3 & -\lambda a_3 & 0 \end{pmatrix} : a_3 \in \mathbb{R} \text{ and } \delta_{0,c_1} = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In case $\lambda = 0$*, we have the previous set specialized in* $\lambda = 0$*, together with the following 1-parameter familly* $\delta_{c_1}^{(1,0)} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} : c_1 \in \mathbb{R}$.

9 Lie bialgebra structures on $\mathfrak{su}(2)$

1-Cocycles. Consider $\mathfrak{su}(2)$ as the $\mathbb{R}$-span of the following matrices:

$$u = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ v = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ w = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then the Lie brackets verify $[u, v] = w$, $[v, w] = u$, $[w, u] = v$. This is a simple Lie algebra, then every 1-cocycle is a 1-coboundary. If $r = \alpha u \wedge v + \beta v \wedge w + \gamma w \wedge u \in \Lambda^2\mathfrak{su}(2)$, with α, β, $\gamma \in \mathbb{R}$, the 1-cocycle associated to it is $\delta(x) = \mathrm{ad}_x(r) = [x, r]$ for any $x \in \mathfrak{su}(2)$. The Co-Jacobi condition for δ is equivalent to $[r, r] \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$ with $[r, r] = 2(\alpha^2 + \beta^2 + \gamma^2)u \wedge w \wedge v$, so it is satisfied for any r since $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = \Lambda^3\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{su}(2)$. We get

$$\delta(u) = \gamma u \wedge v - \alpha w \wedge u; \ \delta(v) = -\beta u \wedge v + \alpha v \wedge w; \ \delta(w) = -\gamma v \wedge w + \beta w \wedge u;$$

or, in matrix notation $\delta = \begin{pmatrix} \gamma & -\beta & 0 \\ 0 & \alpha & -\gamma \\ -\alpha & 0 & \beta \end{pmatrix}$.

Automorphisms and isomorphism classes. If $U \in \mathrm{SU}(2)$, then conjugation by U gives an automorphism of the Lie algebra $\mathfrak{su}(2)$. If we parametrize such a matrix by

$$U = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$, $a^2+b^2+c^2+d^2 = 1$, we have the automorphism $\phi_U(M) = UMU^{-1}$, where $M \in \mathfrak{su}(2)$. Straightforward computation shows the following:

$$\begin{array}{rclccc} \phi_U(u) & = & (a^2+b^2-c^2-d^2)u & +2(-ac+bd)v & -2(bc+ad)w \\ \phi_U(v) & = & 2(ac+bd)u & +(a^2-b^2-c^2+d^2)v & +2(ab-cd)w \\ \phi_U(w) & = & -2(bc-ad)u & -2(ab+cd)v & +(a^2-b^2+c^2-d^2)w \end{array}$$

and $\delta' := (\phi \wedge \phi)^{-1} \delta \phi$ is given by $\frac{\delta'}{2} =$

$$\left(\begin{array}{c|c|c} \alpha(ab-cd)+\beta(ac+bd) & \alpha(ad+bc)+\gamma(ac-bd) & 0 \\ +\frac{1}{2}\gamma(a^2-b^2-c^2+d^2) & +\frac{1}{2}\beta(-a^2-b^2+c^2+d^2) & \\ \hline 0 & \frac{1}{2}\alpha(a^2-b^2+c^2-d^2) & \alpha(cd-ab)-\beta(bd+ac) \\ & \beta(ad-bc)-\gamma(ab+cd) & +\frac{1}{2}\gamma(-a^2+b^2+c^2-d^2) \\ \hline \frac{1}{2}\alpha(-a^2+b^2-c^2+d^2) & 0 & -\alpha(ad+bc)+\gamma(bd-ac) \\ +\beta(bc-ad)+\gamma(ab+cd) & & +\frac{1}{2}\beta(a^2+b^2-c^2-d^2) \end{array} \right)$$

Suppose $\gamma \neq 0$, and take $b = d = 0$, then

$$\delta' = \begin{pmatrix} \gamma(a^2-c^2)+2\beta ac & 2\gamma ac+\beta(c^2-a^2) & 0 \\ 0 & \alpha(a^2+c^2) & -2\beta ac+\gamma(c^2-a^2) \\ -\alpha(a^2+c^2) & 0 & -2\gamma ac+\beta(a^2-c^2) \end{pmatrix}$$

So, if $c = a\left(\frac{\beta+\sqrt{\beta^2+\gamma^2}}{\gamma}\right)$ then $\gamma' = 0$. Assuming $\gamma = 0$ and letting $a = d = 0$, we get

$$\delta' = \begin{pmatrix} 0 & 2\alpha bc+\beta(-b^2+c^2) & 0 \\ 0 & -\alpha(b^2-c^2)-2\beta bc & 0 \\ \alpha(b^2-c^2)+2\beta bc & 0 & -2\alpha bc+\beta(b^2-c^2) \end{pmatrix}$$

So if $\beta \neq 0$, we can use ϕ_U with $a = d = 0$, $c = b\left(\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{\beta}\right)$ then δ' has $\gamma' = 0$ and $\beta' = 0$. So we may assume from the beginning $\gamma = \beta = 0$. We get

$$\delta' = \alpha \begin{pmatrix} 2(ab-cd) & 2(ad+bc) & 0 \\ 0 & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ (-a^2+b^2-c^2+d^2) & 0 & -2(ad+bc) \end{pmatrix}$$

In case $\alpha = 0$, we have the trivial cobracket. If $\alpha \neq 0$, the condition on the automorphism preserving the condition $\beta = \gamma = 0$ is given by the equations:

$$ab = cd, \; ad = -bc$$

These equations imply $a^2 b = -bc^2$, so if $b \neq 0$, then $a = c = 0$ and δ' is given by

$$\delta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha(b^2+d^2) & 0 \\ \alpha(b^2+d^2) & 0 & 0 \end{pmatrix}$$

but $b^2+d^2=1$, so $\delta_\alpha \cong \delta_{-\alpha}$.
Next we consider the case $b=0$. We have $cd=0=ad$. If $d=0$,

$$\delta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha(a^2+c^2) & 0 \\ -\alpha(a^2+c^2) & 0 & 0 \end{pmatrix} = \delta$$

If $d \neq 0$ then $a=c=0$, $d=\pm 1$ and $\delta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha d^2 & 0 \\ \alpha d^2 & 0 & 0 \end{pmatrix} = -\delta.$

Proposition 9.1. *The set of isomorphism classes of Lie bialgebras with underlying Lie algebra $\mathfrak{su}(2)$ is given by the 1-parameter family*

$$\delta_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & 0 \end{pmatrix} : \alpha \geq 0$$

For each $\alpha \neq 0$, the automorphism group is

$$\left\{ \phi_U = \begin{pmatrix} a^2-c^2 & -2ac & 0 \\ 2ac & a^2-c^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, c \in \mathbb{R},\ a^2+c^2=1 \right\} \cong S^1$$

Remark 9.2. *For any $r = \alpha u\wedge v + \beta v\wedge w + \gamma w\wedge u \in \Lambda^2\mathfrak{su}(2)$, $[r,r] = 2(\alpha^2+\beta^2+\gamma^2)u\wedge w\wedge v \in (\Lambda^3\mathfrak{su}(2))^{\mathfrak{su}(2)}$. We have two possibilities:*

- $[r,r]=0$ *if and only if $r=0$ if and only if $\delta=0$. Notice that the the only triangular structure is the trivial one.*
- $0 \neq [r,r]$, *and so $(\mathfrak{su}(2),\delta)$ with $\delta(-)=\mathrm{ad}_{(-)}(r)$ is almost factorizable.*

It was known (see [A-J]) that $\mathfrak{su}(2)$ admites a unique almost factorizable structure up to scalar multiple. This is in perfect agreement with the results of this section.

10 Lie bialgebra structures on $\mathfrak{sl}(2,\mathbb{R})$

1-Cocycles in $\mathfrak{sl}(2,\mathbb{R})$. The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is usually presented as generated by $\{x,y,h\}$ with brackets $[h,x]=2x$, $[h,y]=-2y$, $[x,y]=h$. But it is convenient to consider instead the ordered basis $\{u,v,w\}$ of $\mathfrak{sl}(2,\mathbb{R})$ and $\{u\wedge v, v\wedge w, w\wedge u\}$ of $\Lambda^2\mathfrak{sl}(2,\mathbb{R})$, where $u=h/2$, $v=(x+y)/2$ and $w=(x-y)/2$ *i.e.*

$$u = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ v = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ w = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In this basis, the brackets are given by $[u,v]=w,\ [v,w]=-u,\ [w,u]=-v$. As in the case $\mathfrak{su}(2)$, the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is simple, then every 1-cocycle is a 1-coboundary and the general considerations made for that case hold here. If $r=\alpha u\wedge v+\beta v\wedge w+\gamma w\wedge u$, then the 1-cocycle associated to it is $\delta(z)=\mathrm{ad}_z(r)=[z,r]$ for any $z\in\mathfrak{sl}(2,\mathbb{R})$. Hence

$$\delta(u) = -\gamma u\wedge v - \alpha w\wedge u,\ \delta(v) = \beta u\wedge v + \alpha v\wedge w,\ \delta(w) = \gamma v\wedge w - \beta w\wedge u,$$

in matrix notation, $\delta = \begin{pmatrix} -\gamma & \beta & 0 \\ 0 & \alpha & \gamma \\ -\alpha & 0 & -\beta \end{pmatrix}$. Co-Jacobi is automatically satisfied.

Automorphisms. In a similar way to the $\mathfrak{su}(2)$ case, if $S \in \mathrm{SL}(2,\mathbb{R})$, then conjugation by S gives the automorphisms of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a,b,c,d \in \mathbb{R}$, $ad-bc=1$, then the automorphism ϕ_S given by $\phi_S(M) = SMS^{-1}$ for any $M \in \mathfrak{sl}(2)$, maps u,v,w to

$$\begin{array}{rcrcccc}
\phi_S(u) & = & (ad+bc)u & + & (cd-ab)v & - & (cd+ab)w \\
\phi_S(v) & = & (bd-ac)u & + & \frac{a^2-b^2-c^2+d^2}{2}v & + & \frac{a^2-b^2+c^2-d^2}{2}w \\
\phi_S(w) & = & -(bd+ac)u & + & \frac{a^2+b^2-c^2-d^2}{2}v & + & \frac{a^2+b^2+c^2+d^2}{2}w
\end{array}$$

In matrix notation,

$$\phi_S = \begin{pmatrix} ad+bc & bd-ac & -(bd+ac) \\ cd-ab & \frac{a^2-b^2-c^2+d^2}{2} & \frac{a^2+b^2-c^2-d^2}{2} \\ -(cd+ab) & \frac{a^2-b^2+c^2-d^2}{2} & \frac{a^2+b^2+c^2+d^2}{2} \end{pmatrix}$$

Let us denote by $\kappa = a^2+b^2+c^2+d^2$, $\kappa_{1,3} = -a^2+b^2-c^2+d^2$, $\kappa_{3,4} = a^2+b^2-c^2-d^2$, $\kappa_{1,4} = -a^2+b^2+c^2-d^2$, etc. *i.e.* the subindices point out the places of the negative signs. If $\delta' := (\phi \wedge \phi)^{-1}\delta\phi$ then it is given by $\delta' =$

$$\begin{pmatrix} \frac{\alpha\kappa_{1,3}+2\beta(ac-bd)+\gamma\kappa_{1,4}}{2} & -\alpha(ab+cd)+\beta(bc+ad)+\gamma(-ab+cd) & 0 \\ 0 & \frac{\alpha\kappa-2\beta(ac+bd)+c\kappa_{3,4}}{2} & -\frac{\alpha\kappa_{1,3}+2\beta(ac-bd)+\gamma\kappa_{1,4}}{2} \\ \frac{-\alpha\kappa+2\beta(ac+bd)-\gamma\kappa_{3,4}}{2} & 0 & \alpha(ab+cd)-\beta(ad+bc)+\gamma(ab-cd) \end{pmatrix}$$

If $a=d$ and $c=-b$, with $ad-bc=a^2+b^2=1$, we get

$$\delta' = \begin{pmatrix} -2\beta ab-\gamma(a^2-b^2) & \beta(a^2-b^2)-2\gamma ab & 0 \\ 0 & \alpha & 2\beta ab+\gamma(a^2-b^2) \\ -\alpha & 0 & -\beta(a^2-b^2)+2\gamma ab \end{pmatrix}$$

Because $a^2+b^2=1$, there exists $\theta \in \mathbb{R}$ such that $a=\cos(\theta)$ and $b=\sin(\theta)$. It follows that

$$\begin{array}{rcl}
\beta' & = & \beta(a^2-b^2)-2\gamma ab = \beta\cos(2\theta)-\gamma\sin(2\theta) \\
\gamma' & = & 2\beta ab+\gamma(a^2-b^2) = \beta\sin(2\theta)+\gamma\cos(2\theta)
\end{array}$$

Namely, the pair (β,γ) transform as a rotation, so we can change it, for example, into $(\sqrt{\beta^2+\gamma^2},0)$. In other words, we can assume that $\gamma=0$ and that $\beta \geq 0$.

Now if $\delta = \begin{pmatrix} 0 & \beta & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & -\beta \end{pmatrix}$ with $\beta \geq 0$, we can take an automorphism with $d=a$, $b=c$ and $ad-bc=a^2-b^2=1$. Such a, b may be writen in the form $a=\cosh\theta$, $b=\sinh\theta$ for a $\theta \in \mathbb{R}$. Under an automorphism like this, δ changes into

$$\delta' = \begin{pmatrix} 0 & \beta(a^2+b^2)-2\alpha ab & 0 \\ 0 & -2\beta ab+\alpha(a^2+b^2) & 0 \\ 2\beta ab-\alpha(a^2+b^2) & 0 & -\beta(a^2+b^2)+2\alpha ab \end{pmatrix}$$

This says, for the coefficients of δ', that $\gamma'=0$,

$$\begin{array}{rcl}
\beta' & = & \beta(a^2+b^2)-2\alpha ab = \beta\cosh(2\theta)-\alpha\sinh(2\theta) \\
\alpha' & = & -2\beta ab+\alpha(a^2+b^2) = -\beta\sinh(2\theta)+\alpha\cosh(2\theta)
\end{array}$$

There are three possibilities:

1. $\beta^2-\alpha^2 > 0$. In this case, we can choose θ such that $\alpha'=0$.

2. $\beta^2-\alpha^2 < 0$. In this case, we can choose θ such that $\beta'=0$.

3. $\beta^2 - \alpha^2 = 0$. In this case $\alpha = \pm\beta$. But the automorphism with $a = d = 0$, $b = 1 = -c$ changes $\alpha' = \alpha$ and $\beta' = -\beta$. So, we can assume $\beta = \alpha$.

Case $\alpha = 0$, $\beta \neq 0$. In this case, under a general automorphism, δ changes into

$$\delta' = \begin{pmatrix} \beta(ac-bd) & \beta(ad+bc) & 0 \\ 0 & -\beta(ac+bd) & -\beta(ac-bd) \\ \beta(ac+bd) & 0 & -\beta(ad+bc) \end{pmatrix}$$

An automorphism preserving $\alpha' = 0 = \gamma'$ must satisfy $ac = 0 = bd$. If $a \neq 0$ then $c = 0$; but $ad - bc = 1$, so $d \neq 0$, $b = 0$ and $d = a^{-1}$; in this case $\delta' = \delta$.

If $a = 0$, then $b \neq 0$ since $ad - bc = 1$; so $d = 0$ and $c = -b^{-1}$. In this case $\delta' = -\delta$. We conclude that β may be chosen up to a sign, so a list of representatives of the isomorphism classes is given by

$$\left\{ \delta_\beta = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix} : \beta > 0 \right\}$$

For each of these, the automorphism group is $\left\{ \phi_S : S = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} ; 0 \neq a \in \mathbb{R} \right\}$. Notice that $\phi_S = \phi_{-S}$, so this group is connected, and, it is isomorphic to $(\mathbb{R}, +)$.

Case $\alpha \neq 0$, $\beta = 0$: Under an automorphism ϕ_S, $\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & 0 \end{pmatrix} \mapsto \delta' =$

$$= \frac{\alpha}{2} \begin{pmatrix} -a^2+b^2-c^2+d^2 & -2(ab+cd) & 0 \\ 0 & a^2+b^2+c^2+d^2 & a^2-b^2+c^2-d^2) \\ -(a^2+b^2-c^2+d^2) & 0 & 2(ab+cd) \end{pmatrix}$$

If $\beta' = \gamma' = 0$ then

$$-a^2 + b^2 - c^2 + d^2 = 0 \tag{4}$$
$$ab + cd = 0 \tag{5}$$
$$ad - bc = 1 \tag{6}$$

If $c = 0$, then

$$-a^2 + b^2 + d^2 = 0, \quad ab = 0, \quad ad = 1$$

The last two equations imply $a \neq 0$, $b = 0$, $d = a^{-1}$ and the first one gives $a^4 = 1$ then $a = \pm 1$. In both cases, the automorphism acts trivially.

If $c \neq 0$ we solve $d = -ab/c$ from equation (5) and the other equations transform into

$$\frac{(a^2+c^2)(c^2-b^2)}{c^2} = 0, \quad -b\frac{a^2+c^2}{c} = 1$$

Since $c \neq 0$, the first equation is equivalent to $c^2 = b^2$ then $b = \pm c$. But if $b = c$, the last equation gives $-(a^2 + c^2) = 1$, which is absurd, so $b = -c$, hence $a^2 + c^2 = 1$ and $d = a$. If this is the case, $\delta' = \delta$. The set of isomorphism classes of this type is the 1-parameter family of representatives given by

$$\left\{ \delta_\alpha = \alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} : \alpha \in \mathbb{R},\ \alpha \neq 0 \right\}$$

For each of theses classes, the automorphisms group consists of

$$\left\{\phi_S : S = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}, a^2+c^2=1\right\} = \left\{\begin{pmatrix} a^2-c^2 & -2ac & 0 \\ 2ac & a^2-c^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a^2+c^2=1\right\}$$

Case $\alpha=\beta$: If $\delta = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & -\alpha \end{pmatrix}$ then under an isomorphism Φ_S we get

$$\delta' = \alpha \begin{pmatrix} \frac{-a^2+b^2+2ac-c^2-2bd+d^2}{2} & -2(a-c)(b-d) & 0 \\ 0 & \frac{a^2+b^2-2ac+c^2-2bd+d^2}{2} & \frac{a^2-b^2-2ac+c^2+2bd-d^2}{2} \\ \frac{-a^2-b^2+2ac-c^2+2bd-d^2}{2} & 0 & 2(a-c)(b-d) \end{pmatrix}$$

If one wants to preserve $\alpha'=\beta'$, $\gamma'=0$ then we have two possibilities: $\alpha=0$, or

$$\begin{aligned} -a^2+b^2+2ac-c^2-2bd+d^2 &= 0, \\ a^2+b^2-2ac+c^2-2bd+d^2 &= -2(a-c)(b-d) \end{aligned}$$

which are equivalent to $(b-d)^2 = -(a-c)(b-d)$ and $(a-c)^2 = (c-a)(b-d)$, then $c-a=b-d$, so, setting $d=a-c+b$ we get $\delta' = \alpha(a-c)^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. The condition on a,b,c,d is $ad-bc=1$, then

$$\begin{aligned} 1 &= ad-bc &&= a(a-c+b)-bc \\ &= a(a+b)-c(a+b) &&= (a+b)(a-c) \end{aligned}$$

So, there is no restriction on the value of $a-c$, except being different from zero. Inside its isomorphism class, α is determined up to positive scalar and it is enough to take $\alpha=\pm1$. Hence, there are only three isomorphism classes of triangular Lie bialgebras on $\mathfrak{sl}(2)$:

$$\delta_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \delta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix};\ \delta_{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Proposition 10.1. *The set of isomorphism classes of Lie bialgebras on $\mathfrak{sl}(2,\mathbb{R})$ are given by the following representatives*

$$\textit{Factorizable: } \delta_\beta = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix} : \beta>0; \quad \delta_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & 0 \end{pmatrix} : \alpha\in\mathbb{R},\ \alpha\neq 0.$$

$$\textit{Triangular: } \delta=0; \quad \delta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}; \quad \delta_{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Besides, the Lie bialgebra corresponding to δ_β verifies $\mathfrak{g}\cong\mathfrak{g}^{cop}$. On the other hand, although $\delta_{-1}=-\delta_1$, the corresponding Lie bialgebras are non isomorhphic.

Remark 10.2. *For any $r=\alpha u\wedge v+\beta v\wedge w+\gamma w\wedge u\in\Lambda^2\mathfrak{sl}(2,\mathbb{R})$, $[r,r]=2(\alpha^2-\beta^2-\gamma^2)u\wedge v\wedge w\in\Lambda^3\mathfrak{sl}(2,\mathbb{R})=(\Lambda^3\mathfrak{sl}(2,\mathbb{R}))^{\mathfrak{sl}(2,\mathbb{R})}$. We have the possibilities:*

- *$[r,r]=0$ if and only if $\alpha^2-\beta^2-\gamma^2=0$. So, unlike the $\mathfrak{su}(2)$ case, there are non trivial triangular structures, explicitly given by $\delta_{\pm1}$ with $\alpha=\beta=\pm1$.*
- *$0\neq[r,r]$, and so $(\mathfrak{sl}(2,\mathbb{R}),\delta)$ with $\delta(-)=\mathrm{ad}_{(-)}(r)$ is factorizable.*

The factorizable structures for $\mathfrak{sl}(2,\mathbb{R})$, up to a scalar multiple, are well known (see [A-J]). This is in perfect agreement with the results of this section.

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